PRINCIPAL VALUES OF THE INTEGRAL FUNCTIONALS OF BROWNIAN MOTION: EXISTENCE, CONTINUITY AND AN EXTENSION OF ITÔ'S FORMULA

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Abstract. Let B be a one-dimensional Brownian motion and $f:\mathbb{R}\to\mathbb{R}$ be a Borel function that is locally integrable on $\mathbb{R}\setminus\{0\}$. We present necessary and sufficient conditions (in terms of the function f) for the existence of the limit

$$\lim_{\varepsilon\downarrow 0}\int_0^t f(B_s)\,I(|B_s|>\varepsilon)\,ds$$

in probability and almost surely. This limit (if it exists) can be called the *principal value* of the integral $\int_0^t f(B_s)\,ds$.

The obtained results are applied to give an extension of Itô's formula with the principal value as the covariation term.

We also show that the principal value defines a continuous additive functional of zero energy.

Key words and phrases. Principal values, extensions of Itô's formula, continuous additive functionals of a Brownian motion, processes of zero energy, Brownian local times, Bessel processes, Bessel Bridges.

1 Introduction

1. Existence of the principal values. Let $T\in\mathbb{R}_+$ and $(B_t)_{t\geq 0}$ be a Brownian motion started at $B_0\in\mathbb{R}$. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable on $\mathbb{R}$ function (notation: $f\in L^1_{loc}(\mathbb{R})$), i.e.

$$\forall M>0,\ \int_{-M}^{M}|f(x)|\,dx<\infty.$$

It follows from the properties of the Brownian local times (see Section 2 below) that in this case we have

$$\int_0^T |f(B_s)|\,ds<\infty\quad\text{a.s.},$$

and therefore, there exists almost surely a Lebesgue integral

$$\int_0^T f(B_s)\,ds. \tag{1.1}$$

Suppose now that f is not locally integrable. Then, by the *Engelbert-Schmidt zero-one law* (see [3]), we have

$$\mathsf{P}\Big\{\int_0^T |f(B_s)|\,ds = \infty\Big\} > 0.$$

So, the Lebesgue integral (1.1) does not exist. However, in some special cases there exists a *principal value* of integral (1.1). Take, for example, $f(x) = |x|^\gamma \operatorname{sgn} x$ with $\gamma > -3/2$. Then, due to the properties of the Brownian local times (see the book [9; Problem 1, p. 72] by K. Itô and H.P. McKean for the case $\gamma = -1$ and the paper [1] by Ph. Biane and M. Yor for the case $\gamma > -3/2$), there exists almost surely a limit

$$\text{v.p.}\int_0^T f(B_s)\,ds := \lim_{\varepsilon\downarrow 0}\int_0^T f(B_s)\,I(|B_s| > \varepsilon)\,ds. \tag{1.2}$$

The question arises: for which functions does limit (1.2) exist? We assume from the outset that $f \in L^1_{loc}(\mathbb{R}\setminus\{0\})$ (i.e. condition (3.1) below is satisfied). So, the integrals in the right-hand side of (1.2) are well defined. We present in Section 3 the necessary and sufficient conditions for the existence of limit (1.2) in probability and almost surely (see Theorems 3.1 and 3.2). The conditions are given in nonrandom terms, i.e. in terms of the function f. We also present an example of the function f for which limit (1.2) exists in probability but does not exist almost surely.

The principal values of the form (1.2) are closely connected with various areas of the stochastic analysis. In particular, they are directly related to the extensions of Itô's formula as well as to the continuous zero-energy additive functionals of a Brownian motion. Connections between the principal values and other topics are described in the paper [21] by T. Yamada. The distributional properties of the principal values for the functions f of some special form are discussed in the book [25; Ch. 10] by M. Yor. Let us also mention the paper [8] by Y. Hu, Z. Shi and the paper [7] by Y. Hu, where various laws of the iterated logarithm are derived for the principal value (1.2) with $f(x) = 1/x$.

2. An extension of Itô's formula. We prove in Section 4 the following extension of Itô's formula (see Theorem 4.1): if φ is absolutely continuous on $\mathbb{R}$, φ' is absolutely continuous on $\mathbb{R}\setminus\{0\}$ and limit (1.2) exists in probability for $f = \varphi''$, then

$$\varphi(B_t) = \varphi(B_0) + \int_0^t \varphi'(B_s)\,dB_s + \frac{1}{2}\alpha L_t^0 + \frac{1}{2}\text{v.p.}\int_0^t \varphi''(B_s)\,ds, \tag{1.3}$$

where α is a constant (specified in Theorem 4.1) and L is the local time of B.

There exist several other extensions of Itô's formula: the Itô-Tanaka-Meyer formula (see, for example, [18; Ch. VI, (1.5)]), the Bouleau-Yor formula (see [2], [23]) and the Föllmer-Protter-Shiryaev formula (see [5]). All these extensions differ in the class of the functions φ to which they can be applied and also in the form of the covariation term. In Section 4, we cite the precise formulations of the above-mentioned extensions and show the relation between these extensions and formula (1.3) (see Figure 2 in Section 4).

We also present an example which shows that formula (1.3) could be useful in the theory of the optimal stopping (see Example 4.2).

The comparison of (1.3) and the Bouleau-Yor formula yields a representation of a principal value as an *integral with respect to the local time* (see Corollary 4.5). The comparison of (1.3) and the Föllmer-Protter-Shiryaev formula yields a representation of a principal value as a *quadratic covariation* (see Corollary 4.4).

3. Properties of the principal values. Using the above extension of Itô's formula, we prove in Section 5 that the process v.p.$\int_0^t f(B_s)\,ds$ (if limit (1.2) exists in probability) has a continuous "in t" version (see Theorem 5.1).

We also prove that this process is an additive functional of a Brownian motion and it has zero energy (see Theorems 5.3, 5.5).

The continuous additive functionals of zero energy are well studied (see, for example, the book [6; Ch. 5] by M. Fukushima, Y. Oshima and M. Takeda). In particular, the paper [16] by Y. Oshima and T. Yamada presents a complete characterization of the continuous zero-energy additive functionals of a Brownian motion.

2 Basic Definitions and Facts

This section contains the known definitions and facts that will be used in the subsequent reasoning.

1. Local times. As above, $T \in \mathbb{R}_+$ and $(B_t)_{t\geq 0}$ is a Brownian motion started at $B_0 \in \mathbb{R}$. There exists a continuous process $(L_T^x)_{x\in\mathbb{R}}$ called the *local time of* B, such that, for any locally integrable function h,

$$\int_0^T h(B_s)\,ds = \int_{\mathbb{R}} L_T^x\,h(x)\,dx \quad \text{a.s.} \tag{2.1}$$

(see [18; Ch. VI, §1]).

As stated in the following proposition, the local time of a Brownian motion is a semimartingale.

Proposition 2.1. *There exist a filtration $(\mathcal{G}_x)_{x\in\mathbb{R}}$ and a $(\mathcal{G}_x)$-adapted process $(\beta_x)_{x\in\mathbb{R}}$ such that*

$$\int_{-\infty}^{\infty} |\beta_x|\,dx < \infty \quad \text{a.s.}$$

and the process

$$L_T^x - \int_{-\infty}^{x} \beta_y\,dy$$

is a $(\mathcal{G}_x)$-local martingale with the quadratic variation $\int_{-\infty}^x 4L_T^y\,dy$. Moreover, β is almost surely continuous at $x = 0$.

For the proof, see [11; Théorème II.1.1].

Proposition 2.2. *Set* $S_T = \sup_{s\le T} B_s$, $I_T = \inf_{s\le T} B_s$. *Then*

$$\mathsf{P}\bigl\{\forall x \in (I_T, S_T),\ L_T^x > 0\bigr\} = 1,$$
$$\mathsf{P}\bigl\{\forall x \notin (I_T, S_T),\ L_T^x = 0\bigr\} = 1.$$

For the proof, see [18; Ch. VI, §2].

Proposition 2.3. *Let* $a \in \mathbb{R}$. *The r.v.* L_T^a *has the same distribution as* $(S_T - |a - B_0|) \vee 0$, *where* $S_T = \sup_{s\le T} B_s$.

For $a = B_0$, this statement follows from P. Lévy's theorem (see [18; Ch. VI, (2.3)]). For $a \ne B_0$, one should apply the same reasoning as in the proof of P. Lévy's theorem (its proof is based on the Skorokhod lemma) to get the desired result.

2. Bessel processes. Let $\delta \ge 0$, $a \ge 0$. The solution of the stochastic differential equation

$$Y_t = a + \delta t + 2\int_0^t \sqrt{|Y_s|}\, dW_s \tag{2.2}$$

is called the *square of a* δ*-dimensional Bessel process started at* a. (Equation (2.2) is known to have a unique solution).

`Notation`. The square of a δ-dimensional Bessel process started at a will be designated as $\mathrm{BESQ}^\delta(a)$. The distribution of a $\mathrm{BESQ}^\delta(a)$ on $[0,t]$ will be denoted as $\mathrm{Q}_a^{\delta,t}$ (it is a measure on $C([0,t])$).

Let $[c,d] \subset \mathbb{R}$. By a $\mathrm{BESQ}^\delta(a)$ on $[c,d]$ we will mean the process obtained from a $\mathrm{BESQ}^\delta(a)$ on $[0, d-c]$ by the shift $t \mapsto t + c$. □

`Remark`. If $\delta \in \mathbb{N}$, then $\mathrm{Q}_a^{\delta,t}$ coincides with the distribution of the process $(\|W_s\|^2)_{s\in[0,t]}$, where $(W_s)_{s\in[0,t]}$ is a δ-dimensional Brownian motion started at a point $W_0 \in \mathbb{R}^\delta$ with $\|W_0\|^2 = a$ (this is a consequence of Itô's formula applied to $\|W\|^2$). □

Proposition 2.4. *Suppose that* $\delta, \eta \in [2, \infty)$ *and* $a > 0$. *Then, for any* $t \ge 0$, $\mathrm{Q}_a^{\delta,t} \sim \mathrm{Q}_a^{\eta,t}$.

The proof of this statement can be found in the papers [22], [24], where the precise form of the density is also given. Another proof of the above Proposition follows from the general theory of change of measure (see [10; Ch. IV, §4b]).

3. Bessel Bridges. Let $(X_s)_{s\in[0,t]}$ be the coordinate process on $C([0,t])$, i.e. $X_s : C([0,t]) \ni x \mapsto x(s)$. We denote by $\mathrm{Q}_{a,b}^{\delta,t}$ $(b \ge 0)$ the regular conditional distribution of $\mathrm{Q}_a^{\delta,t}$ with respect to $\sigma(X_t)$. In other words, for any Borel sets $A \subseteq C([0,t])$ and $D \subseteq [0,\infty)$,

$$\mathrm{Q}_a^{\delta,t}(A \cap \{X_t \in D\}) = \int_D \mathrm{Q}_{a,b}^{\delta,t}(A)\, \mu(db),$$

where $\mu = \mathrm{Law}(X_t | \mathrm{Q}_a^{\delta,t})$. There exists a unique modification of $\mathrm{Q}_{a,b}^{\delta,t}$ such that the map $(a,b) \mapsto \mathrm{Q}_{a,b}^{\delta,t}$ is continuous in the weak topology on probability measures (see [17], [18; Ch. XI, §3]). In what follows, we will always choose such a modification of $\mathrm{Q}_{a,b}^{\delta,t}$.

Definition 2.5. The measure $\mathrm{Q}_{a,b}^{\delta,t}$ is called the *law of the* δ*-dimensional Squared Bessel Bridge from* a *to* b *over* $[0,t]$.

Proposition 2.6. *Let* $(Z_s)_{s\in[0,t]}$ *be a process such that*

$$\mathrm{Law}(Z_s;\ 0\le s\le t)=\mathrm{Q}^{\delta,t}_{a,b}.$$

Then

$$\mathrm{Law}(Z_{t-s};\ 0\le s\le t)=\mathrm{Q}^{\delta,t}_{b,a}.$$

For the proof, see [4].

Proposition 2.7. *Suppose that* $\delta\ge 2$, $a>0$, $b\ge 0$. *Fix* $0\le s<t$. *Then the restrictions of* $\mathrm{Q}^{\delta,t}_{a,b}$ *and* $\mathrm{Q}^{\delta,t}_{a}$ *to* $C([0,s])$ *are equivalent.*

For the proof, see [4].

3 Existence of the Principal Values

1. The results. We will investigate here the existence of limit (1.2) in probability and almost surely. The integrals in the right-hand side of (1.2) are the usual Lebesgue integrals. Using equality (2.1) and Proposition 2.2, one can easily note that these integrals exist almost surely if and only if

$$\forall\, 0<\varepsilon<M<\infty,\quad \int_{\mathbb{R}}|f(x)|\,I(\varepsilon\le|x|\le M)\,dx<\infty. \tag{3.1}$$

Theorem 3.1. *Suppose that* $f\in L^1_{loc}(\mathbb{R}\setminus\{0\})$, *i.e.* f *satisfies condition* (3.1). *Limit* (1.2) *exists in probability if and only if the following conditions are satisfied:*

(i) *there exists a limit*

$$\lim_{\varepsilon\downarrow 0}\int_{-1}^{1}f(x)\,I(|x|>\varepsilon)\,dx; \tag{3.2}$$

(ii) *for the function* $F_+(x)=\int_x^1 f(y)\,dy$, *one has*

$$\int_0^1 F_+^2(x)\,dx<\infty,\qquad \varepsilon F_+^2(\varepsilon)\underset{\varepsilon\downarrow 0}{\longrightarrow}0; \tag{3.3}$$

(iii) *for the function* $F_-(x)=\int_{-1}^x f(y)\,dy$, *one has*

$$\int_{-1}^0 F_-^2(x)\,dx<\infty,\qquad |\varepsilon|F_-^2(\varepsilon)\underset{\varepsilon\uparrow 0}{\longrightarrow}0. \tag{3.4}$$

Theorem 3.2. *Suppose that* $f\in L^1_{loc}(\mathbb{R}\setminus\{0\})$. *Limit* (1.2) *exists almost surely if and only if the following conditions are satisfied:*

(i) *there exists limit* (3.2);

(ii) *the function* $F_+(x)=\int_x^1 f(y)\,dy$ *satisfies condition* (3.3), *and, for any* $\alpha>0$,

$$\int_0^1\frac{1}{x}\exp\left\{\frac{-\alpha x}{\sup_{0<y\le x}y^2F_+^2(y)}\right\}dx<\infty; \tag{3.5}$$

(iii) *the function* $F_-(x)=\int_{-1}^x f(y)\,dy$ *satisfies condition* (3.4), *and, for any* $\alpha>0$,

$$\int_{-1}^0\frac{1}{|x|}\exp\left\{\frac{-\alpha|x|}{\sup_{x\le y<0}y^2F_-^2(y)}\right\}dx<\infty. \tag{3.6}$$

`Remarks`. (i) The conditions for the existence of limit (1.2) depend neither on T nor on B_0.

(ii) Let us consider the function f of the form: $f(x) = |x|^\gamma \operatorname{sgn} x$. If $\gamma > -3/2$, then f satisfies the conditions of Theorem 3.2, and thus, limit (1.2) exists almost surely. If $\gamma \le -3/2$, then f does not satisfy conditions (ii) and (iii) of Theorem 3.1, and thus, limit (1.2) in probability does not exist.

(iii) Condition (3.5) is equivalent to the following one: $W_\varepsilon F_+(\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} 0$, where W is a Brownian motion started at zero (see the paper [13; Proposition 15] by T. Jeulin and M. Yor). In other words, for any $\alpha > 0$, the function $\alpha|F_+|^{-1}$ is an *upper function* of a Brownian motion.

(iv) As pointed out by M. Yor (in a personal discussion), condition (ii) of Theorem 3.1 on its own is equivalent to the existence in probability of the limit

$$\lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 f(x)\,(L_T^x - L_T^0)\,dx,$$

where L is the local time of B. □

Theorem 3.1 leads to

Corollary 3.3. *Suppose that* $f(x) = \frac{g(x)}{x}$, *where* $g \in L^2_{loc}(\mathbb{R})$, *i.e.*

$$\forall M > 0, \quad \int_{-M}^{M} (g(x))^2 dx < \infty.$$

Then limit (1.2) *exists in probability if and only if there exists a limit*

$$\lim_{\varepsilon\downarrow 0} \int_{-1}^{1} \frac{g(x)}{x}\, I(|x| > \varepsilon)\,dx.$$

`Proof`. It follows from Hardy's L^2-inequality (see, for example, [12; Lemme 7]) that, for

$$\widetilde{H}g(x) = \int_x^1 \frac{g(y)}{y}\,dy,$$

we have

$$\int_0^1 \big(\widetilde{H}g(x)\big)^2 dx \le 2\int_0^1 (g(x))^2 dx < \infty,$$
$$\varepsilon\big(\widetilde{H}g(\varepsilon)\big)^2 \xrightarrow[\varepsilon\downarrow 0]{} 0.$$

Thus, f satisfies condition (ii) of Theorem 3.1. In a similar way, we verify that f satisfies condition (iii) of this theorem. Hence, limit (1.2) exists in probability if and only if f satisfies condition (i) of Theorem 3.1. □

2. The proofs. Theorems 3.1 and 3.2 follow from Lemmas 3.4–3.8 given below. The scheme of the proof is illustrated in Figure 1.

$$\exists \lim_{\varepsilon \downarrow 0} \int_0^T f(B_s)\, I(|B_s| > \varepsilon)\, ds$$

$\Updownarrow$ (Lemma 3.4)

$$\exists \lim_{\varepsilon \downarrow 0} \int_{-1}^1 f(t)\, Y_t\, I(|t| > \varepsilon)\, dt$$

$\Updownarrow$ (Lemma 3.5)

$$\exists \lim_{\varepsilon \downarrow 0} \int_{-1}^1 f(t) I(|t| > \varepsilon) dt, \quad \exists \lim_{\varepsilon \downarrow 0} \int_{-1}^1 f(t) W_t I(|t| > \varepsilon) dt, \quad \exists \lim_{\varepsilon \downarrow 0} \int_{-1}^1 f(t) W_t^2 I(|t| > \varepsilon) dt$$

$\Updownarrow$ (Lemma 3.6)

condition (i) is satisfied, $\exists \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^1 f(t) W_t dt$, $\exists \lim_{\varepsilon \downarrow 0} \int_{-1}^{-\varepsilon} f(t) W_t dt$, $\exists \lim_{\varepsilon \downarrow 0} \int_{-1}^1 f(t) W_t^2 I(|t| > \varepsilon) dt$

$\Updownarrow$ (Lemma 3.7)

condition (i) is satisfied, condition (ii) is satisfied, condition (iii) is satisfied, $\exists \lim_{\varepsilon \downarrow 0} \int_{-1}^1 f(t) W_t^2 I(|t| > \varepsilon) dt$

$\Updownarrow$ (Lemma 3.8)

condition (i) is satisfied, condition (ii) is satisfied, condition (iii) is satisfied

Figure 1. The scheme of the proof of Theorems 3.1, 3.2

Here, $(B_t)_{t \ge 0}$ is a Brownian motion on $[0, \infty)$ started at B_0;
$(Y_t)_{t \in [-1,1]}$ is a $\mathrm{BESQ}^2(0)$ on $[-1, 1]$;
$(W_t)_{t \in [-1,1]}$ is a Brownian motion on $[-1, 1]$ with $W_0 = 0$.

Lemma 3.4. *Let* $(Y_t)_{t\in[-1,1]}$ *be a* $\mathrm{BESQ}^2(0)$ *on* $[-1,1]$. *Then limit* (1.2) *exists in probability* (*resp: almost surely*) *if and only if the limit*

$$\lim_{\varepsilon\downarrow 0}\int_{-1}^{1} f(t)\,Y_t\,I(|t|>\varepsilon)\,dt \tag{3.7}$$

exists in probability (*resp: almost surely*).

Proof. Fix $b\in(0,1)$. Set

$$U_t = L_T^t, \qquad -b\le t\le b,$$
$$M_t = L_T^t - \int_{-b}^{t}\beta_s\,ds, \qquad -b\le t\le b,$$

where L is the local time of B and β is given by Proposition 2.1. Set

$$\tau = \inf\bigl\{t\ge -b : U_t\notin[b,1/b] \ \text{ or } \ \beta_t\notin[-1/b,1/b]\bigr\},$$
$$\overline{V}_t = b\vee U_{t\wedge\tau}, \quad t\ge -b.$$

We take here $\inf\emptyset = b$. It follows from the choice of τ that the r.v.

$$\int_{-b}^{\tau}(2-\beta_s)^2\,d\langle M\rangle_s = \int_{-b}^{\tau}(2-\beta_s)^2\,4\overline{V}_s\,ds$$

is bounded. Let us consider the measure Q defined by

$$\frac{d\mathsf{Q}}{d\mathsf{P}} = \exp\biggl\{\int_{-b}^{\tau}(2-\beta_s)\,dM_s - \frac{1}{2}\int_{-b}^{\tau}(2-\beta_s)^2\,d\langle M\rangle_s\biggr\}.$$

Applying Girsanov's theorem (and keeping Proposition 2.1 in mind), we can write

$$\overline{V}_t = \overline{V}_0 + \int_0^{t\wedge\tau}2\,ds + 2\int_0^t\sqrt{\overline{V}_s}\,d\overline{W}_s,$$

where $(\overline{W}_t)_{t\ge -b}$ is a $(\mathcal{G}_t,\mathsf{Q})$-local martingale with $\langle\overline{W}\rangle_t = b+t\wedge\tau$ (the filtration $(\mathcal{G}_t)$ is given by Proposition 2.1). There exist an enlargement $\bigl(\Omega',(\mathcal{G}'_t),\mathsf{Q}'\bigr)$ of $\bigl(\Omega,(\mathcal{G}_t),\mathsf{Q}\bigr)$ and a $(\mathcal{G}'_t,\mathsf{Q}')$-local martingale $(W_t)_{t\ge -b}$ such that $\langle W\rangle_t = b+t$ and $W_t=\overline{W}_t$ for $t\le\tau$. Without the loss of generality, we may assume that $\Omega'=\Omega$, $\mathcal{G}'_t=\mathcal{G}_t$, $\mathsf{Q}'=\mathsf{Q}$. The general theory of stochastic differential equations (see [18; Ch. IX, (3.5)]) guarantees that there exists a unique solution $(V_t)_{t\ge -b}$ of the equation

$$V_t = \overline{V}_0 + \int_0^t 2\,ds + 2\int_0^t\sqrt{|V_s|}\,dW_s. \tag{3.8}$$

Moreover, V is positive. Set $\sigma = \tau\wedge\inf\{t\ge -b : V_t\notin[b,1/b]\}$. For any $t\ge -b$, we have

$$\mathsf{E}(V_{t\wedge\sigma}-\overline{V}_{t\wedge\sigma})^2 = 4\int_0^{t\wedge\sigma}\mathsf{E}\Bigl(\sqrt{V_s}-\sqrt{\overline{V}_s}\Bigr)^2 ds \le \frac{1}{b}\int_0^t\mathsf{E}(V_{s\wedge\sigma}-\overline{V}_{s\wedge\sigma})^2 ds.$$

Applying Gronwall's lemma, we deduce that $V_t=\overline{V}_t$ for $t\le\sigma$. This leads to the equality $\sigma=\tau$. As a result, $V_t=\overline{V}_t$ for $t\le\tau$. Thus, on the set $\{\tau=b\}$ we have: $\forall t\in[-b,b]$, $V_t=\overline{V}_t=U_t$.

The existence of limit (1.2) in probability is equivalent to the following condition: *for any sequence* (a_n, b_n) *such that* $0 < a_n < b_n$ *and* $b_n \to 0$*, one has*

$$\int_{-1}^{1} f_n(t)\, U_t\, dt \xrightarrow[n\to\infty]{\mathsf{P}} 0, \tag{3.9}$$

where $f_n(t) = f(t)\, I(a_n \le |t| \le b_n)$, $U_t = L_T^t$. (We keep formula (2.1) in mind).

Fix a sequence (a_n, b_n) with $0 < a_n < b_n$ and $b_n \to 0$. Let Q^n, V^n and τ^n denote the corresponding objects Q, V and τ constructed for b_n instead of b. Set

$$\widetilde{\mathsf{P}}^n(A) = \mathsf{P}(A \,|\, U_{-b_n} > b_n), \qquad \widetilde{\mathsf{Q}}^n(A) = \mathsf{Q}^n(A \,|\, U_{-b_n} > b_n).$$

It follows from Proposition 2.2 that, for almost every ω in the set $\{U_0 = 0\}$, one has: $\exists \varepsilon = \varepsilon(\omega) > 0 : U_t(\omega) = 0$ on $(-\varepsilon, \varepsilon)$. Thus,

$$\int_{-1}^{1} f_n(t)\, U_t\, dt \xrightarrow[n\to\infty]{\mathsf{P}} 0 \iff \int_{-1}^{1} f_n(t)\, U_t\, dt \xrightarrow[n\to\infty]{\widetilde{\mathsf{P}}^n} 0. \tag{3.10}$$

(The notation $\xi_n \xrightarrow{\mathsf{P}^n} \xi$ means that $\mathsf{P}^n\{|\xi_n - \xi| > \delta\} \to 0$ for any $\delta > 0$). Proposition 2.3, together with the continuity of β at zero, guarantees that

$$\widetilde{\mathsf{P}}^n\{\tau^n = b_n\} \xrightarrow[n\to\infty]{} 1,$$

and therefore,

$$\widetilde{\mathsf{P}}^n\big\{\forall t \in [-b_n, b_n],\ V_t^n = U_t\big\} \xrightarrow[n\to\infty]{} 1.$$

Moreover, $\widetilde{\mathsf{P}}^n \sim \widetilde{\mathsf{Q}}^n$ and $d\widetilde{\mathsf{Q}}^n / d\widetilde{\mathsf{P}}^n \xrightarrow{\widetilde{\mathsf{P}}^n} 1$. Thus,

$$\int_{-1}^{1} f_n(t)\, U_t\, dt \xrightarrow[n\to\infty]{\widetilde{\mathsf{P}}^n} 0 \iff \int_{-1}^{1} f_n(t)\, V_t^n\, dt \xrightarrow[n\to\infty]{\widetilde{\mathsf{P}}^n} 0 \iff \int_{-1}^{1} f_n(t)\, V_t^n\, dt \xrightarrow[n\to\infty]{\widetilde{\mathsf{Q}}^n} 0. \tag{3.11}$$

Now, let $\widetilde{\mathrm{R}}^n = \mathrm{Law}\big(V_t^n;\, -b_n \le t \le b_n \,\big|\, \widetilde{\mathsf{Q}}^n\big)$. The general theory of stochastic differential equations (see [14; (18.10)]), together with (3.8), guarantees that

$$\widetilde{\mathrm{R}}^n = \int_{b_n}^{\infty} \mathrm{R}_y^n\, \widetilde{\mu}^n(dy),$$

where R_y^n is the distribution of a $\mathrm{BESQ}^2(y)$ on $[-b_n, b_n]$ and

$$\begin{aligned} \widetilde{\mu}^n &= \mathrm{Law}\big(V_{-b_n}^n \,\big|\, \widetilde{\mathsf{Q}}^n\big) = \mathrm{Law}\big(V_{-b_n}^n \,\big|\, \widetilde{\mathsf{P}}^n\big) = \mathrm{Law}\big(b_n \vee U_{-b_n} \,\big|\, \widetilde{\mathsf{P}}^n\big) \\ &= \mathrm{Law}\big(b_n \vee (S_T - |b_n + B_0|) \,\big|\, S_T > b_n + |b_n + B_0|\big). \end{aligned} \tag{3.12}$$

(We use here Proposition 2.3).

Let Y be a $\mathrm{BESQ}^2(0)$ on $[-1, 1]$ and $\mathrm{R}^n = \mathrm{Law}(Y_t;\, -b_n \le t \le b_n)$. Then

$$\mathrm{R}^n = \int_0^{\infty} \mathrm{R}_y^n\, \mu^n(dy),$$

where R_y^n is the same as above and $\mu^n = \mathrm{Law}(Y_{-b_n})$. We have $\mu^n = \mathrm{Law}(\|W_{1-b_n}\|^2)$, where $(W_t)_{t\ge 0}$ is a two-dimensional Brownian motion started at zero (see the Remark before Proposition 2.4). This, combined with the explicit form of $\widetilde{\mu}^n$ given by (3.12), shows that

$$\widetilde{\mu}^n(A_n) \to 0 \iff \mu^n(A_n) \to 0$$

for any sequence (A_n) of Borel sets. Consequently,

$$\widetilde{\mathrm{R}}^n(D_n) \to 0 \iff \mathrm{R}^n(D_n) \to 0$$

for any sequence (D_n) of Borel sets. Combining this with (3.10), (3.11), we deduce that (3.9) is equivalent to the condition

$$\int_{-1}^{1} f_n(t)\, Y_t\, dt \xrightarrow[n\to\infty]{\mathsf{P}} 0.$$

Thus, limit (1.2) exists in probability if and only if limit (3.7) exists in probability.

The second part of the Lemma (that deals with the existence of (1.2) and (3.7) almost surely) is proved in a similar way. □

Lemma 3.5. *Let $(Y_t)_{t\in[-1,1]}$ be a $\mathrm{BESQ}^2(0)$ on $[-1,1]$ and $(W_t)_{t\in[-1,1]}$ be a Brownian motion on $[-1,1]$ with $W_0 = 0$ (i.e. $(W_t)_{t\in[0,1]}$ and $(W_{-t})_{t\in[0,1]}$ are independent Brownian motions started at zero). Then limit (3.7) exists in probability (resp: almost surely) if and only if the limits*

$$\lim_{\varepsilon\downarrow 0} \int_{-1}^{1} f(t)\, I(|t| > \varepsilon)\, dt, \tag{3.13}$$

$$\lim_{\varepsilon\downarrow 0} \int_{-1}^{1} f(t)\, W_t\, I(|t| > \varepsilon)\, dt, \tag{3.14}$$

$$\lim_{\varepsilon\downarrow 0} \int_{-1}^{1} f(t)\, W_t^2\, I(|t| > \varepsilon)\, dt \tag{3.15}$$

exist in probability (resp: almost surely).

Proof. Suppose that limit (3.7) exists in probability. Take a sequence (a_n, b_n) such that $0 < a_n < b_n$ and $b_n \to 0$. Set $f_n(t) = f(t)\, I(a_n \le |t| \le b_n)$, $\mathrm{P} = \mathrm{Law}(Y_t;\ -1 \le t \le 1)$ and let $(X_t)_{t\in[-1,1]}$ denote the coordinate process on $C([-1,1])$, i.e. $X_t : C([-1,1]) \ni x \mapsto x(t)$. We have

$$\int_{-1}^{1} f_n(t)\, X_t\, dt \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{3.16}$$

Let P_a $(a \ge 0)$ be a version of the regular conditional distribution of P with respect to the σ-field $\sigma(X_0)$, i.e. for any Borel sets $A \subseteq C([-1,1])$ and $D \subseteq [0,\infty)$,

$$\mathrm{P}(A \cap \{X_0 \in D\}) = \int_D \mathrm{P}_a(A)\, \mu(da),$$

where $\mu = \mathrm{Law}(X_0|\mathrm{P})$. The following properties hold for μ-almost every a:
(A) $\mathrm{Law}(X_t;\ -1 \le t \le 0 \,|\, \mathrm{P}_a)$ is the distribution of the two-dimensional Squared Bessel Bridge from 0 to a over $[-1,0]$;
(B) $\mathrm{Law}(X_t;\ 0 \le t \le 1 \,|\, \mathrm{P}_a)$ is the distribution of a $\mathrm{BESQ}^2(a)$ on $[0,1]$;

(C) the processes $(X_t)_{t\in[-1,0]}$ and $(X_t)_{t\in[0,1]}$ are P_a-independent. (Properties (B), (C) follow from the Markov property of Y).

In view of (3.16), there exist $a > 0$ and a subsequence (n_k) such that conditions (A)–(C) are satisfied and

$$\int_{-1}^{1} f_{n_k}(t)\, X_t\, dt \xrightarrow[k\to\infty]{\mathsf{P}_a} 0. \tag{3.17}$$

Let $(Y_t^1)_{t\in[0,1]}$, $(Y_t^2)_{t\in[0,1]}$ be two independent $\mathrm{BESQ}^2(a)$ on $[0,1]$; let $(\widetilde{Y}_t^1)_{t\in[0,1]}$, $(\widetilde{Y}_t^2)_{t\in[0,1]}$ be two independent $\mathrm{BESQ}^3(a)$ on $[0,1]$. Set

$$Z_t^a = \begin{cases} Y_{-t}^1 & \text{if } t\in[-1,0],\\ Y_t^2 & \text{if } t\in[0,1],\end{cases} \qquad \widetilde{Z}_t^a = \begin{cases} \widetilde{Y}_{-t}^1 & \text{if } t\in[-1,0],\\ \widetilde{Y}_t^2 & \text{if } t\in[0,1],\end{cases}$$

$$\mathsf{Q}_a = \mathrm{Law}(Z_t^a;\ -1\le t\le 1), \qquad \widetilde{\mathsf{Q}}_a = \mathrm{Law}(\widetilde{Z}_t^a;\ -1\le t\le 1).$$

In view of Propositions 2.4, 2.6 and 2.7, the restrictions of P_a, Q_a and $\widetilde{\mathsf{Q}}_a$ to the σ-field $\sigma(X_t;\ -1/2\le t\le 1)$ are equivalent (we recall that conditions (A)–(C) are satisfied for the chosen a). Therefore, (3.17) implies that

$$\int_{-1}^{1} f_{n_k}(t)\, X_t\, dt \xrightarrow[k\to\infty]{\mathsf{Q}_a} 0, \qquad \int_{-1}^{1} f_{n_k}(t)\, X_t\, dt \xrightarrow[k\to\infty]{\widetilde{\mathsf{Q}}_a} 0. \tag{3.18}$$

The measure $\widetilde{\mathsf{Q}}_a$ coincides with the distribution of the process

$$\widetilde{V}_t = (\sqrt{a} + W_t^1)^2 + (W_t^2)^2 + (W_t^3)^2, \tag{3.19}$$

where $(W_t^i)_{t\in[-1,1]}$ $(i = 1,2,3)$ are three independent Brownian motions on $[-1,1]$ with $W_0^i = 0$ (see the Remark before Proposition 2.4). Similarly, Q_a coincides with the distribution of the process

$$V_t = (\sqrt{a} + W_t^1)^2 + (W_t^2)^2,$$

where $(W_t^i)_{t\in[-1,1]}$ $(i = 1,2)$ are the same as in (3.19). Thus, (3.18) is equivalent to:

$$\int_{-1}^{1} f_{n_k}(t)\, \big[(\sqrt{a} + W_t^1)^2 + (W_t^2)^2 + (W_t^3)^2\big]\, dt \xrightarrow[k\to\infty]{\mathsf{P}} 0,$$

$$\int_{-1}^{1} f_{n_k}(t)\, \big[(\sqrt{a} + W_t^1)^2 + (W_t^2)^2\big]\, dt \xrightarrow[k\to\infty]{\mathsf{P}} 0.$$

These conditions are, in turn, equivalent to the following ones:

$$\int_{-1}^{1} f_{n_k}(t)\, (a + 2\sqrt{a}\, W_t)\, dt \xrightarrow[k\to\infty]{\mathsf{P}} 0, \tag{3.20}$$

$$\int_{-1}^{1} f_{n_k}(t)\, (W_t)^2\, dt \xrightarrow[k\to\infty]{\mathsf{P}} 0. \tag{3.21}$$

The integrals in (3.20) are Gaussian r.v. (see [15; Ch. 7, §5]). For the Gaussian r.v., the convergence in probability implies the L^2-convergence. Thus, (3.20) is equivalent to:

$$\int_{-1}^{1} f_{n_k}(t)\, dt \xrightarrow[k\to\infty]{} 0, \tag{3.22}$$

$$\int_{-1}^{1} f_{n_k}(t)\, W_t\, dt \xrightarrow[k\to\infty]{\mathsf{P}} 0. \tag{3.23}$$

So, we have proved that, from any sequence (a_n, b_n) such that $0 < a_n < b_n$ and $b_n \to 0$, one can extract a subsequence (n_k) for which conditions (3.22), (3.23) and (3.21) are satisfied. This means that limits (3.13)–(3.15) exist in probability.

The reverse implication as well as the statement concerning the existence of the limits almost surely are proved in a similar way. □

Lemma 3.6. *Limit* (3.14) *exists in probability* (*resp: almost surely*) *if and only if the limits*

$$\lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 f(t)\, W_t\, dt, \tag{3.24}$$

$$\lim_{\varepsilon\downarrow 0} \int_{-1}^{-\varepsilon} f(t)\, W_t\, dt \tag{3.25}$$

exist in probability (*resp: almost surely*).

`Proof`. We should prove the "only if" assertion. Suppose that limit (3.14) exists in probability. Set

$$\widetilde{W}_t = \begin{cases} W_t & \text{if } t \in [0,1], \\ -W_t & \text{if } t \in [-1,0]. \end{cases}$$

The distributions of $(\widetilde{W}_t)_{t\in[-1,1]}$ and $(W_t)_{t\in[-1,1]}$ coincide, and therefore, there exists a limit in probability

$$\lim_{\varepsilon\downarrow 0} \int_{-1}^1 f(t)\, \widetilde{W}_t\, I(|t| > \varepsilon)\, dt.$$

Furthermore,

$$\int_{-1}^1 f(t)\, W_t\, I(|t| > \varepsilon)\, dt + \int_{-1}^1 f(t)\, \widetilde{W}_t\, I(|t| > \varepsilon)\, dt = 2\int_\varepsilon^1 f(t)\, W_t\, dt.$$

This completes the proof. □

Lemma 3.7. *Limit* (3.24) *exists in probability* (*resp: almost surely*) *if and only if condition* (ii) *of Theorem 3.1* (*resp: condition* (ii) *of Theorem 3.2*) *is satisfied.*

`Proof`. Suppose that limit (3.24) exists in probability. As the integrals in (3.24) are Gaussian r.v. (see [15; Ch. 7, §5]), this limit also exists in L^2. Therefore, the expression

$$\begin{aligned} \mathsf{E}\Big(\int_\varepsilon^1 f(t)\, W_t\, dt\Big)^2 &= \int_\varepsilon^1 \int_\varepsilon^1 s \wedge t\, f(s) f(t)\, ds dt = 2\int_\varepsilon^1 t f(t)\, F_+(t)\, dt \\ &= -\int_\varepsilon^1 t\, \frac{d}{dt}(F_+^2(t))\, dt = \varepsilon F_+^2(\varepsilon) + \int_\varepsilon^1 F_+^2(t)\, dt \end{aligned}$$

converges to a limit as $\varepsilon \to 0$. Consequently,

$$\int_0^1 F_+^2(t)\, dt < \infty \tag{3.26}$$

and there exists $\lim_{\varepsilon\downarrow 0} \varepsilon F_+^2(\varepsilon)$. Condition (3.26) implies that this limit is equal to zero.

Now, suppose that condition (ii) of Theorem 3.1 is satisfied. For any sequence (a_n, b_n) such that $0 < a_n < b_n$ and $b_n \to 0$, we have

$$\mathsf{E}\Big(\int_{a_n}^{b_n} f(t)\, W_t\, dt\Big)^2 = (F_+(a_n) - F_+(b_n))^2 a_n + \int_{a_n}^{b_n} (F_+(x) - F_+(b_n))^2 dx. \tag{3.27}$$

Condition (3.3) guarantees that the right-hand side of (3.27) tends to zero as $n \to \infty$. Since the sequence (a_n, b_n) was chosen arbitrarily, this means that limit (3.24) exists in probability.

Suppose that limit (3.24) exists almost surely. Then this limit also exists in probability, and, by the above reasoning, condition (3.3) is satisfied. By Itô's formula,

$$\int_\varepsilon^1 f(t)\, W_t\, dt = -\int_\varepsilon^1 W_t\, dF_+(t) = W_\varepsilon F_+(\varepsilon) + \int_\varepsilon^1 F_+(t)\, dW_t. \tag{3.28}$$

Set $\widetilde{W}_t = W_{1-t} - W_1$, $\widetilde{F}_+(t) = F_+(1-t)$. The process $(\widetilde{W}_t)_{t\in[0,1]}$ is a Brownian motion, and we have

$$\int_\varepsilon^1 F_+(t)\, dW_t = -\int_0^{1-\varepsilon} \widetilde{F}_+(t)\, d\widetilde{W}_t.$$

As $\int_0^1 \widetilde{F}_+^2(t)\, dt < \infty$, we get

$$\int_0^{1-\varepsilon} \widetilde{F}_+(t)\, d\widetilde{W}_t \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} \int_0^1 \widetilde{F}_+(t)\, d\widetilde{W}_t.$$

This, combined with (3.28), shows that there exists almost surely $\lim_{\varepsilon\downarrow 0} W_\varepsilon F_+(\varepsilon)$. By Blumenthal's zero-one law, this limit is equal to a constant. The symmetry property of a Brownian motion guarantees that this limit equals zero. According to a result of T. Jeulin and M. Yor (see [13; Proposition 15]), the condition $W_\varepsilon F_+(\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} 0$ is equivalent to (3.5).

The last implication (stating that condition (ii) of Theorem 3.2 guarantees the existence of limit (1.2) almost surely) is proved in a similar way. □

Lemma 3.8. *If conditions* (ii) *and* (iii) *of Theorem 3.1* (*resp: conditions* (ii) *and* (iii) *of Theorem 3.2*) *are satisfied, then limit* (3.15) *exists in probability* (*resp: almost surely*).

`Proof`. Suppose that conditions (ii) and (iii) of Theorem 3.1 are satisfied. By Itô's formula, we have

$$\int_\varepsilon^1 f(t)\, W_t^2\, dt = -\int_\varepsilon^1 W_t^2\, dF_+(t) = W_\varepsilon^2 F_+(\varepsilon) + 2\int_\varepsilon^1 F_+(t)\, W_t\, dW_t + \int_\varepsilon^1 F_+(t)\, dt. \tag{3.29}$$

In view of the inequality $\int_0^1 |F_+(t)|\, dt < \infty$, we deduce that the last term in (3.29) converges to a limit as $\varepsilon \to 0$.

Set $\widetilde{W}_t = W_{1-t}$, $\widetilde{F}_+(t) = F_+(1-t)$. The process $(\widetilde{W}_t)_{t\in[0,1]}$ is a semimartingale with the decomposition

$$\widetilde{W}_t = \widetilde{W}_0 - \int_0^t \frac{\widetilde{W}_s}{1-s}\, ds + U_t,$$

where U is a Brownian motion (see [18; Ch. IV, (3.18)]). We have

$$\int_\varepsilon^1 F_+(t)\,W_t\,dW_t = -\int_0^{1-\varepsilon} \widetilde{F}_+(t)\,\widetilde{W}_t\,d\widetilde{W}_t = \int_0^{1-\varepsilon} \frac{\widetilde{F}_+(t)\,\widetilde{W}_t^2}{1-t}\,dt - \int_0^{1-\varepsilon} \widetilde{F}_+(t)\,\widetilde{W}_t\,dU_t. \tag{3.30}$$

In view of the properties

$$\mathsf{E}\int_0^1 \left|\frac{\widetilde{F}_+(t)\,\widetilde{W}_t^2}{1-t}\right| dt = \int_0^1 |\widetilde{F}_+(t)|\,dt < \infty,$$
$$\mathsf{E}\int_0^1 \bigl(\widetilde{F}_+(t)\,\widetilde{W}_t\bigr)^2 dt \le \int_0^1 \widetilde{F}_+^2(t)\,dt < \infty,$$

we deduce that the expression in (3.30) converges almost surely as $\varepsilon \to 0$.

Furthermore,

$$\mathsf{E}|W_\varepsilon^2 F_+(\varepsilon)| = \varepsilon|F_+(\varepsilon)| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0,$$

and hence, the expression in (3.29) converges in probability as $\varepsilon \to 0$. Applying the same reasoning to the integral $\int_{-1}^{-\varepsilon} f(t)\,W_t^2\,dt$, we deduce that limit (3.15) exists in probability.

Suppose now that conditions (ii) and (iii) of Theorem 3.2 are satisfied. Condition (3.5) means that $W_\varepsilon F_+(\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} 0$ that leads to $W_\varepsilon^2 F_+(\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} 0$. Thus, limit (3.29) exists almost surely. Applying the same reasoning to $\int_{-1}^{-\varepsilon} f(t)\,W_t^2\,dt$, we get the existence of (3.15) almost surely. □

3. Comparison of Theorems 3.1 and 3.2. If limit (1.2) exists almost surely, then it exists in probability. The following example shows that the reverse is not true.

Example 3.9. We take a sequence $1 = b_1 > a_1 > b_2 > a_2 \dots$ with $b_n > 0$, $b_n \to 0$ that satisfies some additional properties to be specified below. Set $F(t) = (t\ln^2 1/t)^{-1/2}$ if t does not belong to any of the intervals (a_n, b_n), and set $F(t) = (t\ln\ln 1/t)^{-1/2}$ if t is the middle of an interval (a_n, b_n). We extend F to the remaining points in $(0,1)$ by linearity. Obviously, $\varepsilon F^2(\varepsilon) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0$. We can take points b_n sufficiently close to each other so that condition (3.5) is violated. We can take each a_n sufficiently close to b_n so that $\int_0^1 F^2(t)\,dt < \infty$.

Set

$$f(t) = \begin{cases} 0 & \text{if } t \notin (-1,1), \\ F'(t) & \text{if } t \in (0,1), \\ -F'(-t) & \text{if } t \in (-1,0). \end{cases}$$

Then f satisfies the conditions of Theorem 3.1 while it does not satisfy conditions (3.5), (3.6) of Theorem 3.2. In other words, for this function f, limit (1.2) exists in probability but does not exist almost surely. □

The function f constructed in the above example is highly oscillating. The theorem below shows that, for rather regular functions f, the existence of limit (1.2) in probability implies its existence almost surely.

Theorem 3.10. *Suppose that limit* (1.2) *exists in probability. Moreover, let us assume that the function* $|xF_+(x)|$ *increases on* $(0,\delta)$ *and the function* $|xF_-(x)|$ *decreases on* $(-\delta,0)$ *for some* $\delta > 0$ (*the functions* F_+ *and* F_- *are defined in Theorem 3.1*)*. Then limit* (1.2) *exists almost surely.*

`Proof`. By Theorem 3.1, condition (3.3) is satisfied. Fix $\alpha > 0$. As $\varepsilon F_+^2(\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{} 0$, we have

$$\frac{1}{x}\exp\left\{\frac{-\alpha x}{\sup_{0<y\le x} y^2F_+^2(y)}\right\} = \frac{F_+^2(x)}{xF_+^2(x)}\exp\left\{\frac{-\alpha}{xF_+^2(x)}\right\} \le F_+^2(x)$$

for sufficiently small x. Keeping (3.3) in mind, we deduce that (3.5) is satisfied. Applying the same reasoning to F_-, we get the result. □

4 An Extension of Itô's Formula

1. Itô's formula and its known extensions. Recall that $(B_t)_{t\ge 0}$ denotes a Brownian motion started at $B_0 \in \mathbb{R}$.

Itô's formula states that if $\varphi \in C^2(\mathbb{R})$, then

$$\varphi(B_t) = \varphi(B_0) + \int_0^t \varphi'(B_s)\,dB_s + \frac{1}{2}\int_0^t \varphi''(B_s)\,ds. \tag{4.1}$$

The Itô-Tanaka-Meyer formula (see, for example, [18; Ch. VI, (1.5)]) states that if φ' is a function of bounded variation, then

$$\varphi(B_t) = \varphi(B_0) + \int_0^t \varphi'(B_s)\,dB_s + \frac{1}{2}\int_{\mathbb{R}} L_t^x\,\varphi''(dx), \tag{4.2}$$

where L is the local time of B and φ'' is defined as a signed measure on $\mathbb{R}$ (it is finite on compact intervals).

N. Bouleau and M. Yor proved in [2] that if φ' is locally bounded, then

$$\varphi(B_t) = \varphi(B_0) + \int_0^t \varphi'(B_s)\,dB_s - \frac{1}{2}\int_{\mathbb{R}} \varphi'(x)\,d_xL_t^x, \tag{4.3}$$

where $\int_{\mathbb{R}} \varphi'(x)\,d_xL_t^x$ is the *integral with respect to the local time* (its precise definition is given in [2]).

H. Föllmer, Ph. Protter and A.N. Shiryaev gave in [5] the following extension of Itô's formula. Suppose that $\varphi' \in L^2_{loc}(\mathbb{R})$, i.e.

$$\forall M > 0,\quad \int_{-M}^{M} (\varphi'(x))^2 dx < \infty.$$

Then, for any $t \ge 0$ and any sequence (t_k^n), $n = 1, 2\dots$ of finite partitions of $[0,t]$ with $\sup_k(t_{k+1}^n - t_k^n) \xrightarrow[n\to\infty]{} 0$, there exists a limit in probability

$$[\varphi'(B), B]_t = \lim_{n\to\infty}\sum_k \big(\varphi'(B_{t_{k+1}^n}) - \varphi'(B_{t_k^n})\big)\big(B_{t_{k+1}^n} - B_{t_k^n}\big)$$

called the *quadratic covariation* of $\varphi'(B)$ and B. Furthermore,

$$\varphi(B_t) = \varphi(B_0) + \int_0^t \varphi'(B_s)\,dB_s + \frac{1}{2}[\varphi'(B), B]_t. \tag{4.4}$$

2. An extension based on the principal values. We present in this paper the following extension of Itô's formula.

Theorem 4.1. *Let φ be an absolutely continuous on $\mathbb{R}$ function such that φ' is absolutely continuous on $\mathbb{R}\setminus\{0\}$. Suppose that*
(i) *there exists a limit $\alpha = \lim_{\varepsilon\downarrow 0}(\varphi'(\varepsilon) - \varphi'(-\varepsilon))$;*
(ii) $\varphi' \in L^2_{loc}(\mathbb{R})$;
(iii) $x(\varphi'(x))^2 \xrightarrow[x\to 0]{} 0$.
Then

$$\varphi(B_t) = \varphi(B_0) + \int_0^t \varphi'(B_s)\,dB_s + \frac{1}{2}\alpha L_t^0 + \frac{1}{2}\text{v.p.}\int_0^t \varphi''(B_s)\,ds, \tag{4.5}$$

where L is the local time of B.

Remark. Obviously, the assumptions of Theorem 4.1 can be reformulated as follows: φ is absolutely continuous on $\mathbb{R}$, φ' is absolutely continuous on $\mathbb{R}\setminus\{0\}$ and limit (1.2) exists in probability for $f = \varphi''$. □

Proof of Theorem 4.1. Take $n \in \mathbb{N}$ and set

$$\begin{aligned} f_n(x) &= \varphi''(x)\,I(|x| > 1/n), \\ F_n(x) &= \begin{cases} \varphi'(x) & \text{if } |x| \ge 1/n, \\ \varphi'(1/n) & \text{if } 0 < x < 1/n, \\ \varphi'(-1/n) & \text{if } -1/n < x < 0, \end{cases} \\ \varphi_n(x) &= \varphi(0) + \int_0^x F_n(y)\,dy, \\ \alpha_n &= F_n(1/n) - F_n(-1/n). \end{aligned}$$

Applying formula (4.2) (and keeping Proposition 2.1 in mind), we get

$$\varphi_n(B_t) = \varphi_n(B_0) + \int_0^t F_n(B_s)\,dB_s + \frac{1}{2}\alpha_n L_t^0 + \frac{1}{2}\int_0^t f_n(B_s)\,ds. \tag{4.6}$$

Condition (ii) guarantees that, for any $M > 0$,

$$\int_{-M}^{M} (F_n(x) - \varphi'(x))^2\,dx \xrightarrow[n\to\infty]{} 0. \tag{4.7}$$

Consequently, for any $M > 0$,

$$\mathsf{E}\int_0^t (F_n(B_s) - \varphi'(B_s))^2\, I(|B_s| \le M)\,ds \Big(= \int_{-M}^{M} (F_n(x) - \varphi'(x))^2\, \mathsf{E}L_t^x\,dx \Big) \xrightarrow[n\to\infty]{} 0.$$

(We use Proposition 2.3 to estimate $\mathsf{E}L_t^x$). This leads to

$$\int_0^t F_n(B_s)\,dB_s \xrightarrow[n\to\infty]{\mathsf{P}} \int_0^t \varphi'(B_s)\,dB_s.$$

Property (4.7) implies that the sequence (φ_n) converges uniformly to φ. Condition (i) means that $\alpha_n \to \alpha$. Finally, Theorem 3.1 shows that

$$\int_0^t f_n(B_s)\,ds \xrightarrow[n\to\infty]{\mathsf{P}} \text{v.p.}\int_0^t \varphi''(B_s)\,ds.$$

Passing to the limit in (4.6), we get (4.5). □

`Remark`. Suppose that φ satisfies the conditions of Theorem 4.1. Set $\psi(x) = \varphi(x)+\beta x^+$, where $\beta \in \mathbb{R}$, $x^+ = x \vee 0$. Then ψ satisfies the conditions of Theorem 4.1, and

$$\psi'(x) = \varphi'(x) + \beta I(x>0), \qquad \psi''(x) = \varphi''(x).$$

Combining formula (4.5) with the equality

$$\beta B_t^+ = \beta B_0^+ + \int_0^t \beta I(B_s>0)\,dB_s + \frac{1}{2}\beta L_t^0$$

(this equality is a particular case of (4.2)), we get

$$\psi(B_t) = \psi(B_0) + \int_0^t \psi'(B_s)\,dB_s + \frac{1}{2}(\alpha+\beta)L_t^0 + \frac{1}{2}\text{v.p.}\int_0^t \psi''(B_s)\,ds.$$

This shows that the term involving L_t^0 in (4.5) is essential. On the other hand, one can get rid of this term by adding a function of the form βx^+ to the original function φ. □

3. Comparison of different extensions. The Itô-Tanaka-Meyer formula (4.2) is more general than Itô's formula (4.1) in the sense that the class of functions to which it can be applied is greater than the class of functions to which Itô's formula can be applied. The Bouleau-Yor formula (4.3) is, in turn, more general than the Itô-Tanaka-Meyer formula (4.2) while the Föllmer-Protter-Shiryaev formula (4.4) is the most general one.

The place of (4.5) in this hierarchy is between Itô's formula and the formula of Föllmer-Protter-Shiryaev. The generality of (4.5) cannot be compared with the generality of formulas (4.2) and (4.3). Figure 2 illustrates the relation between different extensions of Itô's formula.

Formula (4.5) is useful in the case where a function φ "behaves well everywhere except for one point". It is illustrated by the following examples.

Example 4.2. Suppose that $\varphi \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{0\})$. Such functions often arise in connection with the *smooth fit condition* in the theory of the optimal stopping (see [19], [20; Ch. VIII, 2, §2a]). Obviously, formula (4.5) is applicable to such functions φ (note that $\alpha = 0$ in this case).

On the other hand, Itô's formula and the Itô-Tanaka-Meyer formula may not be applicable to φ. Indeed, suppose that φ' has unbounded variation in any neighborhood of zero. Then $\varphi'' \notin L^1_{loc}(\mathbb{R})$, and the integrals

$$\int_0^t \varphi''(B_s)\,ds, \qquad \int_{\mathbb{R}} \varphi''(x)\,L_t^x\,dx$$

are not defined (this is easily seen from equality (2.1) and Proposition 2.2). □

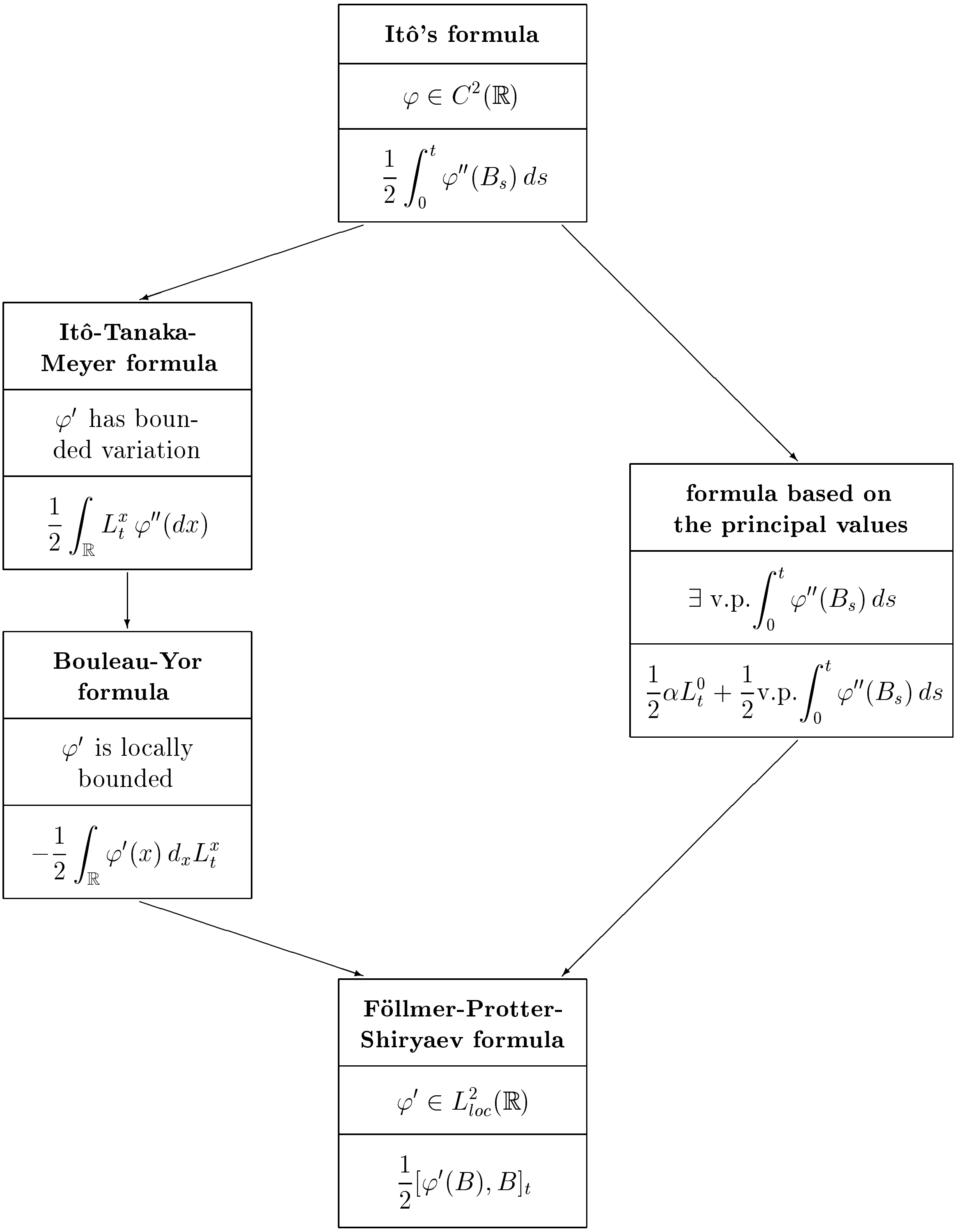

Figure 2. The relation between different extensions of Itô's formula

Each extension of Itô's formula is represented by a box. The arrows indicate the scope of generality of different extensions. The centre line in each box shows the class of functions to which the corresponding extension can be applied. The lower line in each box shows the form of the covariation term for the corresponding extension.

Example 4.3. Take $\varphi(x) = |x|^\gamma \operatorname{sgn} x$ with $1/2 < \gamma < 1$. Then formula (4.5) is applicable to φ while neither the Itô-Tanaka-Meyer formula nor the Bouleau-Yor formula can be applied to φ. □

The comparison of different extensions of Itô's formula allows us to give several representations of the principal value.

Corollary 4.4. *Suppose that f satisfies the conditions of Theorem 3.1. Then there exists a primitive F of the function f (defined separately on $(0,\infty)$ and on $(-\infty,0)$) such that*

$$\lim_{\varepsilon\downarrow 0}(F(\varepsilon) - F(-\varepsilon)) = 0. \tag{4.8}$$

For this F, we have

$$\text{v.p.}\int_0^t f(B_s)\,ds = [F(B), B]_t. \tag{4.9}$$

`Proof`. The existence of a primitive F satisfying (4.8) follows from condition (i) of Theorem 3.1. Let φ be a primitive of F. Combining formulas (4.4) and (4.5), we get the desired result. □

Corollary 4.5. *Suppose that f satisfies the conditions of Theorem 3.1. Let F be a primitive of the function f that satisfies condition* (4.8)*. Suppose that F is locally bounded. Then*

$$\text{v.p.}\int_0^t f(B_s)\,ds = -\int_{\mathbb{R}} F(x)\,d_x L_t^x.$$

`Proof`. This statement follows from equalities (4.3) and (4.5) taken together. □

5 Properties of the Principal Values

Throughout this section, we assume that f satisfies the conditions of Theorem 3.1, i.e. for each $t \geq 0$, there exists

$$\text{v.p.}\int_0^t f(B_s)\,ds. \tag{5.1}$$

We will study here the properties of this process "in t".

1. Continuity. There exists an absolutely continuous on $\mathbb{R}$ function φ such that φ' is absolutely continuous on $\mathbb{R}\setminus\{0\}$ and $\varphi'' = f$. Applying Theorem 4.1, we get the following statement.

Theorem 5.1. *Process* (5.1) *has a continuous version.*

2. Energy. If the function f is not locally integrable, then process (5.1) does not have finite variation. However, in any case it is a process of zero energy.

Definition 5.2. A process $(Z_t)_{t\geq 0}$ has *zero energy* if for any $t \geq 0$ and any sequence (t_k^n), $n = 1, 2\ldots$ of finite partitions of $[0,t]$ with $\sup_k(t_{k+1}^n - t_k^n) \xrightarrow[n\to\infty]{} 0$, one has

$$\sum_k \left(Z_{t_{k+1}^n} - Z_{t_k^n}\right)^2 \xrightarrow[n\to\infty]{\mathsf{P}} 0.$$

Theorem 5.3. *Process* (5.1) *has zero energy.*

`Proof`. It is shown in [5; (3.45)] that, for $F \in L^2_{loc}(\mathbb{R})$, the quadratic covariation $[F(B), B]$ is a process of zero energy. Taking (4.9) into account, we get the result. $\square$

3. Additivity. Let P_x denote the distribution of a Brownian motion started at $x \in \mathbb{R}$. Let $(X_t)_{t\geq 0}$ be the coordinate process on $C(\mathbb{R}_+)$ and $(\mathcal{F}_t)_{t\geq 0}$ be the canonical filtration on $C(\mathbb{R}_+)$, i.e. $\mathcal{F}_t = \sigma(X_s;\, s \leq t)$. Finally, $(\theta_t)_{t\geq 0}$ denotes a family of shifts defined by

$$\theta_t : C(\mathbb{R}_+) \ni (x(s))_{s\geq 0} \longmapsto (x(t+s))_{s\geq 0} \in C(\mathbb{R}_+).$$

Definition 5.4. A *continuous additive functional of a Brownian motion* is a continuous $(\mathcal{F}_t)$-adapted process $(Z_t)_{t\geq 0}$ on $C(\mathbb{R}_+)$ such that

$$Z_{t+s} = Z_s + Z_t \circ \theta_s \quad \mathrm{P}_x\text{-a.s.}$$

for any $t, s \geq 0$, $x \in \mathbb{R}$.

Theorem 5.5. *There exists a continuous additive functional Z of a Brownian motion such that*

$$Z_t = \text{v.p.} \int_0^t f(X_s)\, ds \quad \mathrm{P}_x\text{-a.s.}$$

for any $t \geq 0$, $x \in \mathbb{R}$. Moreover, Z has zero energy with respect to each P_x.

`Proof`. Let F be a primitive of the function f that satisfies condition (4.8). Let φ be a primitive of F. According to Theorem 4.1, we have

$$\text{v.p.} \int_0^t f(X_s)\, ds = 2\varphi(X_t) - 2\varphi(X_0) - 2\int_0^t F(X_s)\, dX_s \quad \mathrm{P}_x\text{-a.s.} \tag{5.2}$$

for any $t \geq 0$ and $x \in \mathbb{R}$. It follows directly from the definition that the right-hand side of (5.2) is an additive functional of a Brownian motion. The second part of the statement follows from Theorem 5.3. $\square$

`Remark`. Y. Oshima and T. Yamada proved in [16] that any continuous zero-energy additive functional of a Brownian motion can be represented as

$$\varphi(X_t) - \varphi(X_0) - \int_0^t \varphi'(X_s)\, dX_s,$$

where φ is an absolutely continuous function with $\varphi' \in L^2_{loc}(\mathbb{R})$. $\square$

4. Convergence to the principal value. If f satisfies the conditions of Theorem 3.1, then, for any $T \geq 0$,

$$\int_0^T f(B_s)\, I(|B_s| > \varepsilon)\, ds \xrightarrow[\varepsilon\downarrow 0]{\mathsf{P}} \text{v.p.} \int_0^T f(B_s)\, ds.$$

We will now prove that the convergence also holds in a stronger sense.

Definition 5.6. A sequence $(Z_t^n)_{t\geq 0}$ of random processes converges to a process $(Z_t)_{t\geq 0}$ *in probability uniformly on compact intervals* if for each $t \geq 0$,

$$\sup_{s\leq t} |Z_s^n - Z_s| \xrightarrow[n\to\infty]{\mathsf{P}} 0.$$

This will be denoted as $Z_t^n \xrightarrow{\text{u.p.}} Z_t$.

Theorem 5.7. *Suppose that f satisfies the conditions of Theorem 3.1. Then*

$$\int_0^t f(B_s)\, I(|B_s| > \varepsilon)\, ds \xrightarrow[\varepsilon\downarrow 0]{\text{u.p.}} \text{v.p.}\int_0^t f(B_s)\, ds.$$

Proof. Let φ be a function such that $\varphi'' = f$. It follows from the proof of Theorem 4.1 that each term in (4.6) (except for $\int_0^t f_n(B_s)\, ds$) converges in probability uniformly on compact intervals to the corresponding term in (4.5). So, the convergence also holds for the term $\int_0^t f_n(B_s)\, ds$. This yields the desired result. □

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References

[1] *Ph. Biane, M. Yor.* Valeurs principales associées aux temps locaux browniens.// Bull. Sci. Math. **111** (1987), p. 23–101.

[2] *N. Bouleau, M. Yor.* Sur la variation quadratique des temps locaux de certaines semi-martingales.// C. R. Acad. Sci. Paris, Série I, **292** (1981), p. 491–494.

[3] *H.J. Engelbert, W. Schmidt.* On the behaviour of certain functionals of the Wiener process and applications to stochastic differential equations.// Lecture Notes in Control and Information Sciences, **36** (1981), p. 47–55.

[4] *P.J. Fitzsimmons, J.W. Pitman, M. Yor.* Markovian Bridges: construction, Palm interpretation and splicing.// Seminar on Stochastic Processes, 1992. Birkhäuser, 1993, p. 101–134.

[5] *H. Föllmer, Ph. Protter, A.N. Shiryaev.* Quadratic covariation and an extension of Itô's formula.// Bernoulli, **1** (1995), No. 1/2, p. 149–170.

[6] *M. Fukushima, Y. Oshima, M. Takeda.* Dirichlet forms and symmetric Markov processes. Walter de Gruyter, 1994.

[7] *Y. Hu.* The laws of Chung and Hirsch for Cauchy's principal values related to Brownian local times.// Electronic Journal of Probability, **5** (2000), No. 10, p. 1–16.

[8] *Y. Hu, Z. Shi.* An iterated logarithm law for Cauchy's principal value of Brownian local times.// Exponential functionals and principal values related to Brownian motion. Part B. Biblioteca de la Revista Matemática Iberoamericana, 1997.

[9] *K. Itô, H.P. McKean.* Diffusion processes and their sample paths. Springer, 1965.

[10] *J. Jacod, A.N. Shiryaev.* Limit theorems for stochastic processes. Springer, 1987.

[11] *T. Jeulin.* Application de la théorie du grossissement à l'étude des temps locaux browniens.// Lecture Notes in Mathematics, **1118** (1985), p. 197–304.

[12] *T. Jeulin, M. Yor.* Inégalité de Hardy, semimartingales et faux-amis.// Lecture Notes in Mathematics, **721** (1979), p. 332–359.

[13] *T. Jeulin, M. Yor.* Filtration des ponts browniens et équations différentielles stochastiques linéaires.// Lecture Notes in Mathematics, **1426** (1989), p. 227–265.

[14] *O. Kallenberg.* Foundations of modern probability. Springer, 1997.

[15] *R.S. Liptser, A.N. Shiryaev.* Statistics of random processes, I and II. Springer, 1977 and 1978.

[16] *Y. Oshima, T. Yamada.* On some representations of continuous additive functionals of locally zero energy.// J. Math. Soc. Japan, **36** (1984), No. 2, p. 315–339.

[17] *J.W. Pitman, M. Yor.* A decomposition of Bessel Bridges.// Z.W. **59** (1982), p. 425–457.

[18] *D. Revuz, M. Yor.* Continuous martingales and Brownian motion. Springer, 1994.

[19] *A.N. Shiryaev.* Optimal stopping rules. Springer, 1978.

[20] *A.N. Shiryaev.* Essentials of stochastic finance. World Scientific, 1998.

[21] *T. Yamada.* Principal values of Brownian local times and their related topics.// Itô's stochastic calculus and probability theory. Springer, 1996, p. 413–422.

[22] *M. Yor.* Loi de l'indice du lacet brownien et distribution de Hartman-Watson.// Z.W. **53** (1980), p. 71–95.

[23] *M. Yor.* Sur la transformée de Hilbert des temps locaux browniens et une extension de la formule d'Itô.// Lecture Notes in Mathematics, **920** (1982), p. 238–247.

[24] *M. Yor.* On some exponential functionals of Brownian motion.// Advances in Applied Probability, **24** (1992), No. 3, p. 509–531.

[25] *M. Yor.* Some aspects of Brownian motion. Part II: Some recent martingale problems. Birkhäuser, 1997.