

SEPARATING TIMES FOR MEASURES ON FILTERED SPACES

A.S. Cherny*, M.A. Urusov**

** Moscow State University,
Faculty of Mechanics and Mathematics,
Department of Probability Theory,
119992 Moscow, Russia.
E-mail: cherny@mech.math.msu.su*

*** Moscow State University,
Faculty of Mechanics and Mathematics,
Department of Probability Theory,
119992 Moscow, Russia.
E-mail: urusov@mech.math.msu.su*

Abstract. We introduce the notion of a *separating time* for a pair of measures \mathbb{P} and $\tilde{\mathbb{P}}$ on a filtered space. This notion is convenient for describing the mutual arrangement of \mathbb{P} and $\tilde{\mathbb{P}}$ from the viewpoint of the absolute continuity and singularity.

Furthermore, we find the explicit form of the separating time for the case, where \mathbb{P} and $\tilde{\mathbb{P}}$ are distributions of Lévy processes, solutions of stochastic differential equations, and distributions of Bessel processes. The obtained results yield, in particular, the criteria for the local absolute continuity, absolute continuity, and singularity of \mathbb{P} and $\tilde{\mathbb{P}}$.

Key words and phrases. Separating time, local absolute continuity, absolute continuity, singularity, Lévy processes, stochastic differential equations, Bessel processes.

1 Introduction

Let \mathbb{P} and $\tilde{\mathbb{P}}$ be probability measures on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)})$. We introduce the notion of a *separating time* for \mathbb{P} and $\tilde{\mathbb{P}}$ (see Definition 2.4). Informally, the separating time is a stopping time, before which \mathbb{P} and $\tilde{\mathbb{P}}$ are equivalent and after which \mathbb{P} and $\tilde{\mathbb{P}}$ are singular. The properties such as the local absolute continuity, absolute continuity, and singularity of \mathbb{P} and $\tilde{\mathbb{P}}$ are easily expressed in terms of the separating time (see Lemma 2.7). The notion of the separating time is mostly convenient in the case, where \mathbb{P} and $\tilde{\mathbb{P}}$ are in a general position, i.e. they are neither locally equivalent nor singular.

In Section 3, we find the explicit form of the separating time for the case, where \mathbb{P} and $\tilde{\mathbb{P}}$ are distributions of Lévy processes (see Theorem 3.1). The criteria for the local absolute continuity, absolute continuity, and singularity of \mathbb{P} and $\tilde{\mathbb{P}}$ are obtained as corollaries of these results. Such criteria are already known (see [10], [11]; earlier related results can be found in [9], [16], [17]).

In Section 4, we find the explicit form of the separating time for the case, where \mathbb{P} and $\tilde{\mathbb{P}}$ are distributions of the solutions of stochastic differential equations (abbreviated below as SDEs); see Theorem 4.7. As a corollary, we obtain the criteria for the local absolute continuity, absolute continuity and singularity of \mathbb{P} and $\tilde{\mathbb{P}}$. Similar results for more general

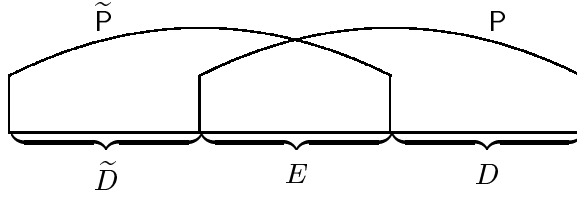


Figure 1. Mutual arrangement of a pair of measures on a measurable space

SDEs can be found in [7; Ch. IV, §4b], where they are obtained using the theory of Hellinger processes. We consider here a more particular case (only homogeneous SDEs), but in this case we obtain more complete results.

In Section 5, we find the explicit form of the separating time for the case, where P and \tilde{P} are distributions of Bessel processes (see Theorem 5.1). Let us emphasize that the Bessel process is a solution of a certain SDE, but the results of Section 4 cannot be applied since this SDE is singular (this notion is introduced in [4]).

The Appendix contains some facts related to the qualitative behaviour of SDE solutions. This is needed for the proofs of Corollaries 4.8, 4.9, and 4.10, but is not needed to understand the results of Section 4.

2 Separating Times

2.1. Mutual arrangement of a pair of measures on a measurable space. Let P and \tilde{P} be probability measures on a measurable space (Ω, \mathcal{F}) . The following result is well known.

Proposition 2.1. *There exists a decomposition $\Omega = E \sqcup D \sqcup \tilde{D}$, $E, D, \tilde{D} \in \mathcal{F}$ such that $\tilde{P} \sim P$ on the set E and $P(\tilde{D}) = \tilde{P}(D) = 0$ (here “ \sqcup ” denotes the disjoint union). This decomposition is unique P, \tilde{P} -a.s.*

Remarks. (i) For the above decomposition, we have $\tilde{P} \sim P$ on E and $\tilde{P} \perp P$ on E^c (here E^c denotes the complement to E). Such a decomposition is also unique P, \tilde{P} -a.s.

(ii) The sets E, D, \tilde{D} from Proposition 2.1 can be obtained by the formulas:

$$\tilde{D} = \left\{ \frac{dP}{dQ} = 0, \frac{d\tilde{P}}{dQ} > 0 \right\}, E = \left\{ \frac{dP}{dQ} > 0, \frac{d\tilde{P}}{dQ} > 0 \right\}, D = \left\{ \frac{dP}{dQ} > 0, \frac{d\tilde{P}}{dQ} = 0 \right\},$$

where $Q = \frac{P + \tilde{P}}{2}$.

The result of Proposition 2.1 is illustrated by Figure 1.

2.2. Mutual arrangement of a pair of measures on a filtered space. Let (Ω, \mathcal{F}) be a measurable space endowed with a right-continuous filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$, i.e. $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$. Recall that the σ -field \mathcal{F}_τ (τ is a stopping time) is defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for any } t \in [0, \infty)\}. \quad (2.1)$$

(In particular, $\mathcal{F}_\infty = \mathcal{F}$.)

Let P and \tilde{P} be probability measures on \mathcal{F} . As usually, P_τ (resp: \tilde{P}_τ) denotes the restriction of P (resp: \tilde{P}) to \mathcal{F}_τ .

Let us add a point δ to $[0, \infty]$ in such a way that $\delta > \infty$.

Definition 2.2. An *extended stopping time* is a map $T : \Omega \rightarrow [0, \infty] \cup \{\delta\}$ such that $\{T \leq t\} \in \mathcal{F}_t$ for any $t \in [0, \infty]$.

The following theorem is an analog of Proposition 2.1 for a filtered space. A similar statement is proved in [8; Lemma 5.2].

Theorem 2.3. (i) *There exists an extended stopping time S such that for any stopping time τ ,*

$$\tilde{\mathbb{P}}_\tau \sim \mathbb{P}_\tau \text{ on the set } \{\tau < S\}, \quad (2.2)$$

$$\tilde{\mathbb{P}}_\tau \perp \mathbb{P}_\tau \text{ on the set } \{\tau \geq S\}. \quad (2.3)$$

(ii) *If S' is another extended stopping time with these properties, then $S' = S$ $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s.*

Proof. (i) Set $\mathbb{Q} = \frac{\mathbb{P} + \tilde{\mathbb{P}}}{2}$. Let $(Z_t)_{t \in [0, \infty]}$ and $(\tilde{Z}_t)_{t \in [0, \infty]}$ denote the density processes of \mathbb{P} and $\tilde{\mathbb{P}}$ with respect to \mathbb{Q} (we set $Z_\infty = \frac{d\mathbb{P}}{d\mathbb{Q}}$, $\tilde{Z}_\infty = \frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}$). Let $(\overline{\mathcal{F}}_t)$ denote the \mathbb{Q} -completion of the filtration (\mathcal{F}_t) . Then the $(\overline{\mathcal{F}}_t, \mathbb{Q})$ -martingales Z and \tilde{Z} have the modifications, whose all trajectories are right-continuous and have left limits. The time

$$\overline{S} = \overline{\inf}\{t \in [0, \infty] : Z_t = 0 \text{ or } \tilde{Z}_t = 0\}$$

($\overline{\inf}$ is the same as \inf except that $\overline{\inf}\emptyset = \delta$) is an extended $(\overline{\mathcal{F}}_t)$ -stopping time. According to [7; Ch. I, Lemma 1.19], there exists an extended (\mathcal{F}_t) -stopping time S such that $S = \overline{S}$ \mathbb{Q} -a.s. It follows from [7; Ch. III, Lemma 3.6] that $Z_t \tilde{Z}_t = 0$ on the stochastic interval $[S, \infty]$ \mathbb{Q} -a.s. Consequently, for any (\mathcal{F}_t) -stopping time τ , we have $Z_\tau \tilde{Z}_\tau = 0$ \mathbb{Q} -a.s. on $\{\tau \geq S\}$. The equality

$$\frac{d\mathbb{P}_\tau}{d\mathbb{Q}_\tau} = \mathbb{E}_\mathbb{Q} \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \middle| \mathcal{F}_\tau \right) = \mathbb{E}_\mathbb{Q}(Z_\infty | \mathcal{F}_\tau) = Z_\tau$$

and the analogous equality for $\frac{d\tilde{\mathbb{P}}_\tau}{d\mathbb{Q}_\tau}$ complete the proof.

(ii) Proposition 2.1 implies that for any stopping time τ , the sets $\{\tau \geq S\}$ and $\{\tau \geq S'\}$ coincide $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s. This yields the desired statement (one needs to consider only the deterministic τ). \square

Definition 2.4. A *separating time* for \mathbb{P} and $\tilde{\mathbb{P}}$ is an extended stopping time that satisfies (2.2) and (2.3) for all stopping times τ .

Remark. It is seen from the proof of Theorem 2.3 (ii) that in defining the separating time one may use only the deterministic τ .

Corollary 2.5. (i) *There exists an extended stopping time S such that for any stopping time τ ,*

$$\tilde{\mathbb{P}}_\tau \ll \mathbb{P}_\tau \text{ on the set } \{\tau < S\}, \quad (2.4)$$

$$\tilde{\mathbb{P}}_\tau \perp \mathbb{P}_\tau \text{ on the set } \{\tau \geq S\}. \quad (2.5)$$

(ii) *If S' is another extended stopping time with these properties, then $S' = S$ $\tilde{\mathbb{P}}$ -a.s.*

Definition 2.6. A *time separating $\tilde{\mathbb{P}}$ from \mathbb{P}* is an extended stopping time that satisfies (2.4) and (2.5) for any stopping time τ .

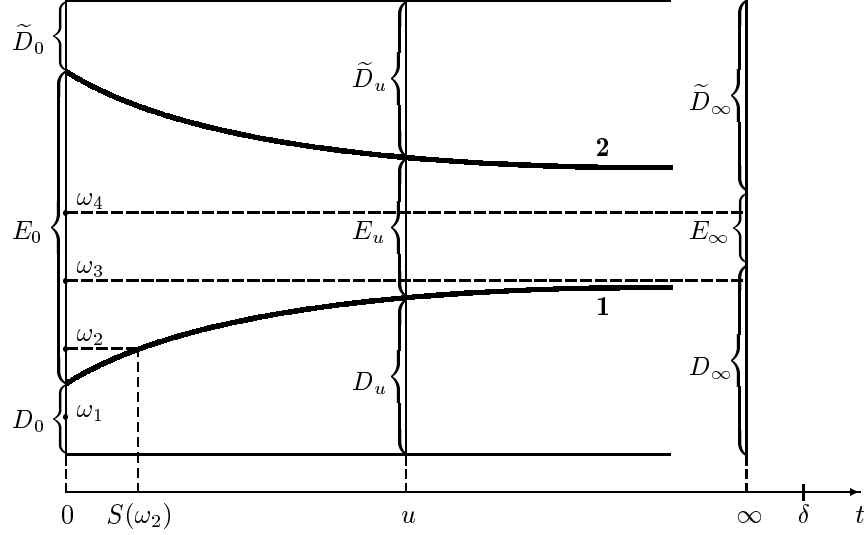


Figure 2. Mutual arrangement of a pair of measures on a filtered space (here $S(\omega_1) = 0$, $S(\omega_3) = \infty$, $S(\omega_4) = \delta$)

Clearly, a separating time for \mathbb{P} and $\tilde{\mathbb{P}}$ is also a time separating $\tilde{\mathbb{P}}$ from \mathbb{P} . The reverse is not true since the former time is unique $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s., while the latter time is unique only $\tilde{\mathbb{P}}$ -a.s.

Informally, Theorem 2.3 states that the measures \mathbb{P} and $\tilde{\mathbb{P}}$ are equivalent up to a random time S and become singular at a time S . The equality $S = \delta$ means that \mathbb{P} and $\tilde{\mathbb{P}}$ never become singular, i.e. they are equivalent up to infinity. Thus, the knowledge of the separating time yields the knowledge of the mutual arrangement of \mathbb{P} and $\tilde{\mathbb{P}}$. This is illustrated by the following result. Its proof is straightforward.

Lemma 2.7. *Let S be a separating time for \mathbb{P} and $\tilde{\mathbb{P}}$. Then*

- (i) $\tilde{\mathbb{P}} \ll^{\text{loc}} \mathbb{P} \iff S \geq \infty \quad \tilde{\mathbb{P}}\text{-a.s.};$
- (ii) $\tilde{\mathbb{P}} \sim^{\text{loc}} \mathbb{P} \iff S \geq \infty \quad \mathbb{P}, \tilde{\mathbb{P}}\text{-a.s.};$
- (iii) $\tilde{\mathbb{P}} \ll \mathbb{P} \iff S = \delta \quad \tilde{\mathbb{P}}\text{-a.s.};$
- (iv) $\tilde{\mathbb{P}} \sim \mathbb{P} \iff S = \delta \quad \mathbb{P}, \tilde{\mathbb{P}}\text{-a.s.};$
- (v) $\tilde{\mathbb{P}} \perp \mathbb{P} \iff S \leq \infty \quad \mathbb{P}, \tilde{\mathbb{P}}\text{-a.s.} \iff S \leq \infty \quad \mathbb{P}\text{-a.s.}$

Remark. Other types of the mutual arrangement of \mathbb{P} and $\tilde{\mathbb{P}}$ are also easily expressed in terms of the separating time. For example, for any $t \in [0, \infty]$,

$$\tilde{\mathbb{P}}_t \perp \mathbb{P}_t \iff S \leq t \quad \mathbb{P}, \tilde{\mathbb{P}}\text{-a.s.} \iff S \leq t \quad \mathbb{P}\text{-a.s.}$$

The mutual arrangement of \mathbb{P} and $\tilde{\mathbb{P}}$ is illustrated by Figure 2. In this figure, the measure $\tilde{\mathbb{P}}$ “lies above” curve 1; the measure \mathbb{P} “lies below” curve 2. The decomposition $\Omega = E_t \sqcup D_t \sqcup \tilde{D}_t$ of Proposition 2.1 for the measurable space (Ω, \mathcal{F}_t) , is obtained by drawing a vertical line corresponding to the time t . Figure 2 shows three decompositions of this type: for $t = 0$, for $t = u \in (0, \infty)$, and for $t = \infty$.

The separating time for \mathbb{P} and $\tilde{\mathbb{P}}$ is illustrated as follows. If $\omega \in D_0 \sqcup \tilde{D}_0$, then $S(\omega) = 0$ (see $\omega = \omega_1$ in Figure 2). If $\omega \in E_0$, then $S(\omega)$ is the time, at which the horizontal line

drawn through the point ω crosses curves 1 or 2 (see $\omega = \omega_2$ in Figure 2). If this line crosses neither curve 1 nor curve 2, then $S = \infty$ in the case $\omega \in D_\infty \sqcup \tilde{D}_\infty$ (see $\omega = \omega_3$ in Figure 2) and $S = \delta$ in the case $\omega \in E_\infty$ (see $\omega = \omega_4$ in Figure 2).

3 Separating Times for Lévy Processes

Let $D([0, \infty), \mathbb{R}^d)$ denote the space of the càdlàg functions $[0, \infty) \rightarrow \mathbb{R}^d$. Let X denote the canonical process on this space, i.e $X_t(\omega) = \omega(t)$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \sigma(X_s; s \in [0, t + \varepsilon])$ and set $\mathcal{F} = \bigvee_{t \in [0, \infty)} \mathcal{F}_t$. In what follows, (\cdot, \cdot) denotes the scalar product in \mathbb{R}^d and $\|\cdot\|$ denotes the Euclidean norm.

Let \mathbb{P} and $\tilde{\mathbb{P}}$ be the distributions of Lévy processes with the characteristics (b, c, ν) and $(\tilde{b}, \tilde{c}, \tilde{\nu})$. This means that for any $t \in [0, \infty)$ and $\lambda \in \mathbb{R}^d$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} e^{i(\lambda, X_t)} &= \exp \left\{ t \left[i(\lambda, b) - \frac{1}{2}(\lambda, c\lambda) \right. \right. \\ &\quad \left. \left. + \int_{\mathbb{R}^d} (e^{i(\lambda, x)} - 1 - i(\lambda, x)I(\|x\| \leq 1)) \nu(dx) \right] \right\}, \end{aligned}$$

where $b \in \mathbb{R}^d$, c is a symmetric positively definite $d \times d$ matrix, and ν is a measure on $\mathcal{B}(\mathbb{R}^d)$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (\|x\|^2 \wedge 1) \nu(dx) < \infty$. For further information on Lévy processes, see [1], [14], [18; Ch. III, § 1b].

The following theorem yields an explicit form of the separating time for \mathbb{P} and $\tilde{\mathbb{P}}$. It easily follows from the results of [15].

Theorem 3.1. *The separating time S for \mathbb{P} and $\tilde{\mathbb{P}}$ has the following form.*

- (i) *If $\mathbb{P} = \tilde{\mathbb{P}}$, then $S = \delta$ $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s.*
- (ii) *If $\mathbb{P} \neq \tilde{\mathbb{P}}$ and*

$$c = \tilde{c}, \tag{3.1}$$

$$\int_{\mathbb{R}^d} \left(\sqrt{\frac{d\nu}{d(\nu + \tilde{\nu})}} - \sqrt{\frac{d\tilde{\nu}}{d(\nu + \tilde{\nu})}} \right)^2 d(\nu + \tilde{\nu}) < \infty, \tag{3.2}$$

$$b - \tilde{b} - \int_{\{\|x\| \leq 1\}} x d(\nu - \tilde{\nu}) \in \mathfrak{N}(c), \tag{3.3}$$

where $\mathfrak{N}(c) = \{cx : x \in \mathbb{R}^d\}$, then

$$S = \inf\{t \in [0, \infty) : \Delta X_t \neq 0, \Delta X_t \notin E\} \quad \mathbb{P}, \tilde{\mathbb{P}}\text{-a.s.}$$

(we set $\inf \emptyset = \infty$), where $E \in \mathcal{B}(\mathbb{R}^d)$ is a set such that $\tilde{\nu} \sim \nu$ on E and $\tilde{\nu} \perp \nu$ on the complement to E .

- (iii) *If any of conditions (3.1)–(3.3) is violated, then $S = 0$ $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s.*

Remark. If (3.2) is true, then $\int_{\{\|x\| \leq 1\}} \|x\| d\|\nu - \tilde{\nu}\| < \infty$, where $\|\nu - \tilde{\nu}\|$ is the total variance of the signed measure $\nu - \tilde{\nu}$ (see [14; Remark 33.3] or [15; Lemma 2.18]). Consequently, the integral in (3.3) is well defined if condition (3.2) is true.

Theorem 3.1, combined with Lemma 2.7, yields the following corollary. This result is known (see [10], [11]).

Corollary 3.2. (i) *Either $\tilde{\mathbb{P}} = \mathbb{P}$ or $\tilde{\mathbb{P}} \perp \mathbb{P}$.*

(ii) *We have $\tilde{\mathbb{P}} \stackrel{\text{loc}}{\ll} \mathbb{P}$ if and only if conditions (3.1)–(3.3) and the condition $\tilde{\nu} \ll \nu$ are satisfied.*

(iii) *We have $\tilde{\mathbb{P}}_0 \perp \mathbb{P}_0$ if and only if any of conditions (3.1)–(3.3) is violated.*

4 Separating Times for Solutions of SDEs

4.1. Basic definitions. We will consider one-dimensional SDEs of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0, \quad (4.1)$$

where b and σ are Borel functions $\mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}$.

Definition 4.1. A *solution* of (4.1) is a pair (Y, B) of continuous adapted processes on a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, \infty)}, \mathbb{Q})$ such that

- i) B is a $(\mathcal{G}_t, \mathbb{Q})$ -Brownian motion;
- ii) for any $t \in [0, \infty)$,

$$\int_0^t (|b(Y_s)| + \sigma^2(Y_s))ds < \infty \quad \mathbb{Q}\text{-a.s.};$$

- iii) for any $t \in [0, \infty)$,

$$Y_t = x_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s \quad \mathbb{Q}\text{-a.s.}$$

Remark. A solution in the sense of Definition 4.1 is sometimes called a *weak solution*.

In what follows, it will be convenient for us to treat a solution as a measure on the space $C([0, \infty))$ of continuous functions. Let X denote the canonical process on $C([0, \infty))$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \sigma(X_s; s \in [0, t + \varepsilon])$ and set $\mathcal{F} = \bigvee_{t \in [0, \infty)} \mathcal{F}_t$.

Definition 4.2. A *solution* of (4.1) is a probability measure \mathbb{P} on \mathcal{F} such that

- i) $\mathbb{P}(X_0 = x_0) = 1$;
- ii) for any $t \in [0, \infty)$,

$$\int_0^t (|b(X_s)| + \sigma^2(X_s))ds < \infty \quad \mathbb{P}\text{-a.s.};$$

- iii) the process

$$M_t = X_t - \int_0^t b(X_s)ds, \quad t \in [0, \infty)$$

is an $(\mathcal{F}_t, \mathbb{P})$ -local martingale with the quadratic variation

$$\langle M \rangle_t = \int_0^t \sigma^2(X_s)ds, \quad t \in [0, \infty).$$

The following statement (see [4; Theorem 1.24]) shows the relationship between Definitions 4.1 and 4.2.

Proposition 4.3. (i) *Let (Y, B) be a solution of (4.1) in the sense of Definition 4.1. Set $\mathbb{P} = \text{Law}(Y_t; t \in [0, \infty))$. Then \mathbb{P} is a solution of (4.1) in the sense of Definition 4.2.*

(ii) *Let \mathbb{P} be a solution of (4.1) in the sense of Definition 4.2. Then there exist a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, \infty)}, \mathbb{Q})$ and a pair of processes (Y, B) on this space such that (Y, B) is a solution of (4.1) in the sense of Definition 4.1 and $\text{Law}(Y_t; t \in [0, \infty)) = \mathbb{P}$.*

4.2. Exploding solutions. Definitions 4.1 and 4.2 do not include the exploding solutions. However, we will need to consider them. Let us introduce some notations.

Let us add a point Δ to the real line and let $C_\Delta([0, \infty))$ denote the space of functions $f : [0, \infty) \rightarrow \mathbb{R} \cup \{\Delta\}$ with the property: there exists a time $\zeta(f) \in [0, \infty]$ such that f is continuous on $[0, \zeta(f))$, $f = \Delta$ on $[\zeta(f), \infty)$, and if $0 < \zeta(f) < \infty$, then $\lim_{t \uparrow \zeta(f)} f(t) = \infty$ or $\lim_{t \uparrow \zeta(f)} f(t) = -\infty$. The time $\zeta(f)$ is called the *explosion time of f* . Below in this section, X will denote the canonical process on $C_\Delta([0, \infty))$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \sigma(X_s; s \in [0, t + \varepsilon])$ and set $\mathcal{F} = \bigvee_{t \in [0, \infty)} \mathcal{F}_t$. Let ζ denote the explosion time of the process X .

The next definition is a generalization of Definition 4.2 to the case of exploding solutions.

Definition 4.4. A *solution* of (4.1) is a probability measure \mathbb{P} on \mathcal{F} such that

- i) $\mathbb{P}(X_0 = x_0) = 1$;
- ii) for any $t \in [0, \infty)$ and $n \in \mathbb{N}$ such that $n > |x_0|$,

$$\int_0^{t \wedge T_n} (|b(X_s)| + \sigma^2(X_s)) ds < \infty \quad \mathbb{P}\text{-a.s.},$$

where $T_n = \inf\{t \in [0, \infty) : X_t = n \text{ or } X_t = -n\}$ (we set $\inf \emptyset = \infty$);

- iii) for any $n \in \mathbb{N}$ such that $n > |x_0|$, the process

$$M_t^n = X_{t \wedge T_n} - \int_0^{t \wedge T_n} b(X_s) ds, \quad t \in [0, \infty)$$

is an $(\mathcal{F}_t, \mathbb{P})$ -local martingale with the quadratic variation

$$\langle M^n \rangle_t = \int_0^{t \wedge T_n} \sigma^2(X_s) ds, \quad t \in [0, \infty).$$

Clearly, if \mathbb{P} is a solution of (4.1) in the sense of Definition 4.4 and $\zeta = \infty$ \mathbb{P} -a.s., then the restriction of \mathbb{P} to $C([0, \infty))$ is a solution of (4.1) in the sense of Definition 4.2. Conversely, if \mathbb{P} is a solution of (4.1) in the sense of Definition 4.2, then there exists a unique extension of the measure \mathbb{P} to $C_\Delta([0, \infty))$ that is a solution of (4.1) in the sense of Definition 4.4.

Definition 4.5. A Borel function $f : \mathbb{R} \rightarrow [0, \infty)$ is *locally integrable at a point $a \in [-\infty, \infty]$* if there exists a neighborhood U of a such that $\int_U f(x) dx < \infty$. (A neighborhood of ∞ is a ray of the form (x, ∞) .) Notation: $f \in L_{\text{loc}}^1(a)$.

A function f is *locally integrable on a set $A \subseteq [-\infty, \infty]$* if f is locally integrable at each point of this set. Notation: $f \in L_{\text{loc}}^1(A)$.

Below we will use the following result (see [6]).

Proposition 4.6 (Engelbert, Schmidt). *Suppose that the coefficients b and σ of (4.1) satisfy the conditions:*

$$\sigma(x) \neq 0 \quad \forall x \in \mathbb{R}, \tag{4.2}$$

$$\frac{1 + |b|}{\sigma^2} \in L_{\text{loc}}^1(\mathbb{R}). \tag{4.3}$$

Then, for any starting point $x_0 \in \mathbb{R}$, there exists a unique solution of (4.1) in the sense of Definition 4.4.

4.3. Explicit form of the separating time. We will use the notations \mathcal{F} , \mathcal{F}_t , X , and ζ introduced in Subsection 4.2.

Consider the SDEs

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0, \quad (4.4)$$

$$dX_t = \tilde{b}(X_t)dt + \tilde{\sigma}(X_t)dB_t, \quad X_0 = x_0 \quad (4.5)$$

with the same starting point x_0 . Let us assume that conditions (4.2), (4.3) and the similar conditions for \tilde{b} , $\tilde{\sigma}$ are satisfied.

Set

$$\rho(x) = \exp\left\{-\int_0^x \frac{2b(y)}{\sigma^2(y)} dy\right\}, \quad x \in \mathbb{R}, \quad (4.6)$$

$$s(x) = \int_0^x \rho(y)dy, \quad x \in \mathbb{R}, \quad (4.7)$$

$$s(\infty) = \lim_{x \rightarrow \infty} s(x), \quad (4.8)$$

$$s(-\infty) = \lim_{x \rightarrow -\infty} s(x). \quad (4.9)$$

Similarly, we define $\tilde{\rho}$, \tilde{s} , $\tilde{s}(\infty)$, and $\tilde{s}(-\infty)$ through \tilde{b} and $\tilde{\sigma}$. Let μ_L denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

We will say that a point $x \in \mathbb{R}$ is *good* if there exists a neighborhood U of x such that $\sigma^2 = \tilde{\sigma}^2$ μ_L -a.e. on U and $(b - \tilde{b})^2/\sigma^4 \in L^1_{\text{loc}}(x)$. We will say that the point ∞ is *good* if all the points belonging to $[x_0, \infty)$ are good and

$$s(\infty) < \infty, \quad (4.10)$$

$$(s(\infty) - s) \frac{(b - \tilde{b})^2}{\rho\sigma^4} \in L^1_{\text{loc}}(\infty). \quad (4.11)$$

We will say that the point $-\infty$ is *good* if all the points belonging to $(-\infty, x_0]$ are good and

$$s(-\infty) > -\infty, \quad (4.12)$$

$$(s - s(-\infty)) \frac{(b - \tilde{b})^2}{\rho\sigma^4} \in L^1_{\text{loc}}(-\infty). \quad (4.13)$$

Let A denote the complement to the set of good points in $[-\infty, \infty]$. Clearly, A is closed in $[-\infty, \infty]$. We will use the notation

$$A^\varepsilon = \{x \in [-\infty, \infty]: \rho(x, A) < \varepsilon\},$$

where $\rho(x, y) = |\arctg x - \arctg y|$, $x, y \in [-\infty, \infty]$ (we set $\emptyset^\varepsilon = \emptyset$).

The main result of this subsection is the following theorem. We do not give its proof since it is rather long. The proof will be given in another paper by the same authors.

Theorem 4.7. *Suppose that b , σ , \tilde{b} , $\tilde{\sigma}$ satisfy conditions (4.2) and (4.3). Let \mathbf{P} and $\tilde{\mathbf{P}}$ denote the solutions of (4.4) and (4.5) in the sense of Definition 4.4. Then the separating time S for \mathbf{P} and $\tilde{\mathbf{P}}$ has the following form.*

(i) *If $\mathbf{P} = \tilde{\mathbf{P}}$, then $S = \delta$ $\mathbf{P}, \tilde{\mathbf{P}}$ -a.s.*

(ii) *If $\mathbf{P} \neq \tilde{\mathbf{P}}$, then*

$$S = \sup_n \overline{\inf} \{t \in [0, \infty): X_t \in A^{1/n}\} \quad \mathbf{P}, \tilde{\mathbf{P}}\text{-a.s.},$$

where $\overline{\inf}$ is the same as \inf except that $\overline{\inf} \emptyset = \delta$.

Remarks. (i) Let us explain the structure of S in case (ii). Let α denote the “bad” point that is closest to x_0 from the right side, i.e.

$$\alpha = \begin{cases} \inf\{x : x \in [x_0, \infty] \cap A\} & \text{if } [x_0, \infty] \cap A \neq \emptyset, \\ \Delta & \text{if } [x_0, \infty] \cap A = \emptyset. \end{cases}$$

Let us consider the “hitting time of α ”:

$$U = \begin{cases} \delta & \text{if } \alpha = \Delta, \\ \delta & \text{if } \alpha = \infty \text{ and } \overline{\lim}_{t < \zeta} X_t < \infty, \\ \zeta & \text{if } \alpha = \infty \text{ and } \overline{\lim}_{t < \zeta} X_t = \infty, \\ T_\alpha & \text{if } \alpha < \infty, \end{cases}$$

where $T_\alpha = \overline{\inf}\{t \in [0, \infty) : X_t = \alpha\}$. Similarly, let β denote the “bad” point that is closest to x_0 from the left side and let V denote the “hitting time of β ”. Then $S = U \wedge V$ $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s.

(ii) It is clear from Theorem 4.7 and the symmetry between \mathbb{P} and $\tilde{\mathbb{P}}$ that, under the condition $[x_0, \infty) \subseteq [-\infty, \infty] \setminus A$, the pair of conditions (4.10), (4.11) is equivalent to the pair

$$\tilde{s}(\infty) < \infty, \quad (4.14)$$

$$(\tilde{s}(\infty) - \tilde{s}) \frac{(b - \tilde{b})^2}{\tilde{\rho} \tilde{\sigma}^4} \in L_{\text{loc}}^1(\infty). \quad (4.15)$$

A similar remark is true for (4.12), (4.13). \square

Theorem 4.7, combined with Lemma 2.7 and Propositions A.1, A.2, yields several corollaries concerning the mutual arrangement of \mathbb{P} and $\tilde{\mathbb{P}}$. In order to formulate them, let us introduce the conditions:

$$\tilde{s}(\infty) = \infty, \quad (4.16)$$

$$\tilde{s}(\infty) < \infty \text{ and } \frac{\tilde{s}(\infty) - \tilde{s}}{\tilde{\rho} \tilde{\sigma}^2} \notin L_{\text{loc}}^1(\infty), \quad (4.17)$$

$$\tilde{s}(\infty) < \infty \text{ and } (\tilde{s}(\infty) - \tilde{s}) \frac{(b - \tilde{b})^2}{\tilde{\rho} \tilde{\sigma}^4} \in L_{\text{loc}}^1(\infty). \quad (4.18)$$

Condition (4.16) means that the paths of the canonical process X under the measure $\tilde{\mathbb{P}}$ do not tend to ∞ as $t \rightarrow \infty$. Condition (4.17) means that the paths of the canonical process X with the positive $\tilde{\mathbb{P}}$ -probability tend to ∞ as $t \rightarrow \infty$, but do not explode into ∞ (i.e. the explosion time for them is ∞). Condition (4.18) is the pair (4.14), (4.15). Similarly, we introduce the conditions at $-\infty$:

$$\tilde{s}(-\infty) = -\infty, \quad (4.19)$$

$$\tilde{s}(-\infty) > -\infty \text{ and } \frac{\tilde{s} - \tilde{s}(-\infty)}{\tilde{\rho} \tilde{\sigma}^2} \notin L_{\text{loc}}^1(-\infty), \quad (4.20)$$

$$\tilde{s}(-\infty) > -\infty \text{ and } (\tilde{s} - \tilde{s}(-\infty)) \frac{(b - \tilde{b})^2}{\tilde{\rho} \tilde{\sigma}^4} \in L_{\text{loc}}^1(-\infty). \quad (4.21)$$

Corollary 4.8. *Under the assumptions of Theorem 4.7, we have $\tilde{\mathbb{P}} \ll_{\text{loc}} \mathbb{P}$ if and only if the conditions*

$$\sigma^2 = \tilde{\sigma}^2 \mu_L\text{-a.e. and } \frac{(b - \tilde{b})^2}{\sigma^4} \in L_{\text{loc}}^1(\mathbb{R}), \quad (4.22)$$

at least one of conditions (4.16)–(4.18), and at least one of conditions (4.19)–(4.21) are satisfied.

Corollary 4.9. *Under the assumptions of Theorem 4.7, we have $\tilde{\mathbb{P}} \ll \mathbb{P}$ if and only if either $\mathbb{P} = \tilde{\mathbb{P}}$; or (4.16), (4.21), (4.22) are satisfied; or (4.18), (4.19), (4.22) are satisfied; or (4.18), (4.21), (4.22) are satisfied.*

Corollary 4.10. *Under the assumptions of Theorem 4.7, we have $\tilde{\mathbb{P}} \perp \mathbb{P}$ if and only if $\tilde{\mathbb{P}} \neq \mathbb{P}$ and $-\infty, \infty \in A$.*

5 Separating Times for Bessel Processes

Consider the SDE

$$dX_t = \gamma dt + 2\sqrt{|X_t|} dB_t, \quad X_0 = x_0$$

with $\gamma \geq 0$, $x_0 \geq 0$. It is known that this SDE has a unique solution Q in the sense of Definition 4.2. Moreover, the measure Q is concentrated on positive functions. A process $(Z_t)_{t \in [0, \infty)}$ with the distribution Q is called the *square of a γ -dimensional Bessel process started at $\sqrt{x_0}$* . The process \sqrt{Z} is called a *γ -dimensional Bessel process started at $\sqrt{x_0}$* . For more information on Bessel processes, see [2], [3], [5], [12], [13; Ch. XI].

Let X denote the canonical process on $C([0, \infty))$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \sigma(X_s; s \in [0, t + \varepsilon])$ and set $\mathcal{F} = \bigvee_{t \in [0, \infty)} \mathcal{F}_t$.

Theorem 5.1. *Let \mathbb{P} (resp. $\tilde{\mathbb{P}}$) be the distribution of a γ -dimensional (resp. $\tilde{\gamma}$ -dimensional) Bessel process started at x_0 . Then the separating time S for \mathbb{P} and $\tilde{\mathbb{P}}$ has the following form.*

(i) *If $\mathbb{P} = \tilde{\mathbb{P}}$, then $S = \delta$ $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s.*

(ii) *If $\mathbb{P} \neq \tilde{\mathbb{P}}$, then*

$$S = \inf\{t \in [0, \infty) : X_t = 0\} \quad \mathbb{P}, \tilde{\mathbb{P}}\text{-a.s.}$$

(we set $\inf \emptyset = \infty$).

Proof. We should prove only (ii). Set $T_0 = \inf\{t \in [0, \infty) : X_t = 0\}$. It follows from [2; Theorem 4.1] and the strong Markov property of Bessel processes that $S \leq T_0$ $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s.

Let us prove that $S \geq T_0$ $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s. For $x_0 = 0$, this is obvious, so we assume that $x_0 > 0$. Fix $\varepsilon \in (0, x_0/2)$ and consider the stopping time $T_\varepsilon = \inf\{t \in [0, \infty) : X_t = \varepsilon\}$. Define the map $F_\varepsilon : C([0, \infty)) \rightarrow C([0, \infty))$ by $F_\varepsilon(\omega)(t) = \omega(t \wedge T_\varepsilon(\omega))$ and let \mathbb{P}^ε denote the image of \mathbb{P} under this map. Using Itô's formula, one can check that \mathbb{P}^ε is a solution of the SDE

$$dX_t = \frac{\gamma - 1}{2X_t} I(t \leq T_\varepsilon) dt + I(t \leq T_\varepsilon) dB_t, \quad X_0 = x_0.$$

Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space with a Brownian motion $(W_t)_{t \in [0, \infty)}$. Consider the space $(C([0, \infty)) \times \Omega', \mathcal{F} \times \mathcal{F}', \mathbb{P}^\varepsilon \times \mathbb{P}')$ and let Q^ε be the distribution of the process

$$Z_t = X_t + \int_0^t I(s > T_\varepsilon) dW_s, \quad t \in [0, \infty).$$

Then Q^ε is a solution of the SDE

$$dX_t = \frac{\gamma - 1}{2X_t} I(t \leq T_\varepsilon) dt + dB_t, \quad X_0 = x_0.$$

Similarly, using the measure $\tilde{\mathbb{P}}$, we define the measure $\tilde{\mathbb{Q}}^\varepsilon$ that is a solution of the SDE

$$dX_t = \frac{\tilde{\gamma} - 1}{2X_t} I(t \leq T_\varepsilon) dt + dB_t, \quad X_0 = x_0.$$

Since the drift coefficients $\frac{\gamma-1}{2X_t}I(t \leq T_\varepsilon)$ and $\frac{\tilde{\gamma}-1}{2X_t}I(t \leq T_\varepsilon)$ are bounded, we get by Girsanov's theorem that $\tilde{\mathbb{Q}}^\varepsilon \stackrel{\text{loc}}{\sim} \mathbb{Q}^\varepsilon$. The obvious equalities $\mathbb{P}^\varepsilon = \mathbb{Q}^\varepsilon \circ F_\varepsilon^{-1}$ and $\tilde{\mathbb{P}}^\varepsilon = \tilde{\mathbb{Q}}^\varepsilon \circ F_\varepsilon^{-1}$ yield that $\tilde{\mathbb{P}}^\varepsilon \stackrel{\text{loc}}{\sim} \mathbb{P}^\varepsilon$. One can verify that $\mathbb{P}^\varepsilon|_{\mathcal{F}_{T_{2\varepsilon}}} = \mathbb{P}|_{\mathcal{F}_{T_{2\varepsilon}}}$ and $\tilde{\mathbb{P}}^\varepsilon|_{\mathcal{F}_{T_{2\varepsilon}}} = \tilde{\mathbb{P}}|_{\mathcal{F}_{T_{2\varepsilon}}}$. Consequently, $\tilde{\mathbb{P}}|_{\mathcal{F}_{t \wedge T_{2\varepsilon}}} \sim \mathbb{P}|_{\mathcal{F}_{t \wedge T_{2\varepsilon}}}$ for any $t \in [0, \infty)$. Since $t \in [0, \infty)$ and $\varepsilon \in (0, x_0/2)$ are arbitrary, we get the desired inequality $S \geq T_0$ $\mathbb{P}, \tilde{\mathbb{P}}$ -a.s. The proof is completed. \square

It is known that if $0 \leq \gamma < 2$, then a γ -dimensional Bessel process started at a strictly positive point hits zero with the probability one; if $\gamma \geq 2$, then a γ -dimensional Bessel process started at a strictly positive point never hits zero with the probability one. Theorem 5.1, combined with Lemma 2.7 and these properties, yields

- Corollary 5.2.** (i) *Either $\tilde{\mathbb{P}} = \mathbb{P}$ or $\tilde{\mathbb{P}} \perp \mathbb{P}$.*
(ii) *If $\tilde{\mathbb{P}} \neq \mathbb{P}$ and $x_0 = 0$, then $\tilde{\mathbb{P}}_0 \perp \mathbb{P}_0$.*
(iii) *Let $\tilde{\mathbb{P}} \neq \mathbb{P}$ and $x_0 > 0$. Then $\tilde{\mathbb{P}} \ll \mathbb{P} \iff \tilde{\gamma} \geq 2$.*

This corollary generalizes the result of [2; Theorem 4.1].

Appendix

Here we describe the behaviour of SDE solutions. We will use the notations \mathcal{F} , \mathcal{F}_t , X , and ζ introduced in Subsection 4.2.

Consider SDE (4.1) and assume that conditions (4.2) and (4.3) are satisfied. According to Proposition 4.6, this equation has a unique solution \mathbb{P} in the sense of Definition 4.4. Consider the sets

$$\begin{aligned} A &= \{ \zeta = \infty, \overline{\lim}_{t \rightarrow \infty} X_t = \infty, \underline{\lim}_{t \rightarrow \infty} X_t = -\infty \}, \\ B_+ &= \{ \zeta = \infty, \lim_{t \rightarrow \infty} X_t = \infty \}, \\ C_+ &= \{ \zeta < \infty, \lim_{t \uparrow \zeta} X_t = \infty \}, \\ B_- &= \{ \zeta = \infty, \lim_{t \rightarrow \infty} X_t = -\infty \}, \\ C_- &= \{ \zeta < \infty, \lim_{t \uparrow \zeta} X_t = -\infty \}. \end{aligned}$$

Define ρ , s , $s(\infty)$, $s(-\infty)$ by formulas (4.6)–(4.9).

The statements below follow from [4; Ch. 4].

Proposition A.1. *Either $\mathbb{P}(A) = 1$ or $\mathbb{P}(B_+ \cup B_- \cup C_+ \cup C_-) = 1$.*

Proposition A.2. (i) *If $s(\infty) = \infty$, then $\mathbb{P}(B_+) = \mathbb{P}(C_+) = 0$.*

(ii) *If $s(\infty) < \infty$ and $(s(\infty) - s)/\rho\sigma^2 \notin L_{\text{loc}}^1(\infty)$, then $\mathbb{P}(B_+) > 0$, $\mathbb{P}(C_+) = 0$.*

(iii) *If $s(\infty) < \infty$ and $(s(\infty) - s)/\rho\sigma^2 \in L_{\text{loc}}^1(\infty)$, then $\mathbb{P}(B_+) = 0$, $\mathbb{P}(C_+) > 0$.*

Clearly, Proposition A.2 has its analog for the behaviour at $-\infty$.

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