ON THE STRONG AND WEAK SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS GOVERNING BESSEL PROCESSES

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Abstract. We prove that the δ-dimensional Bessel process ($\delta > 1$) is a strong solution of a stochastic differential equation of the special form. The purpose of this paper is to investigate whether there exist other (weak and strong) solutions of these equations. This leads us to the conclusion that Zvonkin's theorem can not be extended to the stochastic differential equations with an unbounded drift.

Key words and phrases. Stochastic differential equations, weak and strong solutions, pathwise uniqueness and uniqueness in law, Zvonkin's theorem, Bessel processes.

1 Introduction

This paper deals with the stochastic differential equations (abbreviated below as SDE) of the following form:

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s, \quad X_0 \in \mathbb{R}. \tag{1}$$

Here, $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are Borel functions.

For the sake of being precise, we will cite some definitions related to weak and strong solutions of the SDEs. These definitions are taken from [2; Ch. IX, §1].

Definition 1.1. A *solution* of SDE (1) is a pair (X, B) of adapted processes defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ and such that

(i) B is a standard $(\mathcal{F}_t)$-Brownian motion;

(ii) X is continuous and, for each $t \geq 0$,

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s \quad \mathsf{P}\text{-a.s.}$$

Definition 1.2. A solution (X, B) is said to be a *strong solution* if X is adapted to the filtration $(\overline{\mathcal{F}}_t^B)$, i.e. the filtration of B completed with respect to P.

By contrast, a solution that is not strong will be termed a *weak* solution.

Definition 1.3. There is *uniqueness in law* for (1) if whenever (X, B) and $(\widetilde{X}, \widetilde{B})$ are two solutions (which may be defined on different probability spaces) with $X_0 = \widetilde{X}_0$, then the laws of X and $\widetilde{X}$ are equal.

Definition 1.4. There is *pathwise uniqueness* for (1) if whenever (X, B) and $(\widetilde{X}, B)$ are two solutions on the same filtered probability space with $X_0 = \widetilde{X}_0$, then X and $\widetilde{X}$ are indistinguishable.

The following result is due to Yamada and Watanabe (see [3], [2; Ch. IX, §1]). It illustrates the advantages of the pathwise uniqueness.

Proposition 1.5. *Suppose that the pathwise uniqueness holds for* (1). *Then*
(i) *The uniqueness in law holds for* (1).
(ii) *There exists a measurable map*

$$\Phi : (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+))) \longrightarrow (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$$

such that, for any solution (X, B) *of* (1), *we have* $X(\omega) = \Phi(B(\omega))$ *for* P*-a.e.* ω (*i.e. the processes* X *and* $\Phi(B)$ *are indistinguishable*).

The following example is well-known.

Example 1.6. For the SDE

$$X_t = \int_0^t \operatorname{sgn} X_s \, dB_s, \tag{2}$$

the pathwise uniqueness does not hold. Indeed, if (X, B) is a solution of (2), then $(-X, B)$ is also a solution. However, (2) possesses only weak solutions (see [2; Ch. IX, §1]). □

We present in this paper an example of the SDE with $\sigma \equiv 1$, for which there exist two different *strong* solutions (X, B) and $(\widetilde{X}, B)$ with $X_0 = \widetilde{X}_0$. This equation is rather natural as it is satisfied by the Bessel process.

Note that such an example can be constructed only with an unbounded drift b. This follows from the result obtained by Zvonkin (see [4]):

Proposition 1.7. *If* $\sigma \equiv 1$ *and* b *is bounded, then the pathwise uniqueness holds for* (1).

This paper is arranged as follows. In Section 2, we prove that if (X, B) is a strong solution of (1), then X is a measurable functional of B. This statement is applied in Section 3 where we study the SDEs governing Bessel processes. For these SDEs, we investigate the existence of weak and strong solutions as well as the pathwise uniqueness and the uniqueness in law.

2 Strong Solutions

The following theorem illustrates the advantages of the strong solutions.

Theorem 2.1. *Let* (X,B) *be a strong solution of* (1).

(i) *There exists a measurable map*

$$\Phi : (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+))) \longrightarrow (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$$

such that $X(\omega) = \Phi(B(\omega))$ *for* P*-a.e.* ω.

(ii) *If* $\widetilde{B}$ *is a Brownian motion on a filtered probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathsf{P}})$ *and* $\widetilde{X}(\widetilde{\omega}) := \Phi(\widetilde{B}(\widetilde{\omega}))$, *then the pair* $(\widetilde{X}, \widetilde{B})$ *is a strong solution of* (1).

`Proof.` **(i)** Let P^W denote the Wiener measure on $C(\mathbb{R}_+)$ and let x be the coordinate process. Set $\mathcal{G}_t^0 = \sigma(x_s;\, s \le t)$ and let $\mathcal{G}_t$ denote the completion of $\mathcal{G}_t^0$ with respect to P^W (we attach to $\mathcal{G}_t^0$ all the P^W-null sets from $\mathcal{B}(C(\mathbb{R}_+))$). For each $q \in \mathbb{Q}_+$, there exists a $\mathcal{G}_q$-measurable map $\xi_q : C(\mathbb{R}_+) \to \mathbb{R}$ such that $X_q(\omega) = \xi_q(B(\omega))$ for P-a.e. ω. Set

$$A = \big\{x \in C(\mathbb{R}_+) \,|\, \exists \varphi \in C(\mathbb{R}_+) : \forall q \in \mathbb{Q}_+,\, \xi_q(x) = \varphi(q)\big\}.$$

Due to the continuity of X, we have

$$\mathsf{P}^W(A) \ge \mathsf{P}\{\forall q \in \mathbb{Q}_+,\, X_q = \xi_q(B)\} = 1.$$

Thus, $A \in \mathcal{G}_0$.

For each $n \in \mathbb{N}$, the process

$$\Psi_t^{(n)}(x) = \xi_0(x)\, I(t=0) + \sum_{k=0}^{\infty} \xi_{k/2^n}(x)\, I\big(k/2^n < t \le (k+1)/2^n\big)$$

is $(\mathcal{G}_t)$-optional. Hence, the process

$$\Psi_t(x) = I(x \in A)\, \limsup_{n\to\infty} \Psi_t^{(n)}(x)$$

is also $(\mathcal{G}_t)$-optional. It follows from the definition of A that Ψ is continuous in t for each $x \in C(\mathbb{R}_+)$. Let $\mathcal{G}$ denote the completion of $\mathcal{B}(C(\mathbb{R}_+))$ with respect to P^W. Then Ψ is $\mathcal{G}|\mathcal{B}(C(\mathbb{R}_+))$-measurable. Thus, there exists a measurable map

$$\Phi : (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+))) \longrightarrow (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$$

such that Ψ and Φ are P^W-indistinguishable. Moreover, for each $t \ge 0$, $\Phi_t(B) = X_t$ P-a.s. Thus, the processes $\Phi(B)$ and X are indistinguishable.

(ii) It is sufficient to note that, for each $t \ge 0$, the map $\Phi_t : C(\mathbb{R}_+) \to \mathbb{R}$ is $\mathcal{G}_t$-measurable and furthermore, for each $t \ge 0$,

$$\Phi_t(x) = \Phi_0(x) + \int_0^t b(\Phi_s(x))\, ds + \int_0^t \sigma(\Phi_s(x))\, dx_s \quad \mathsf{P}^W\text{-a.s.} \qquad \square$$

3 SDEs Governing Bessel Processes

We begin with some definitions related to Bessel processes. Let us consider the SDE

$$Z_t = Z_0 + \delta t + 2\int_0^t \sqrt{|Z_s|}\,dB_s, \tag{3}$$

where $\delta \geq 0$, $Z_0 \geq 0$. This equation is known to have a unique strong solution (see [2; Ch. IX, §3]). Moreover, this solution is nonnegative.

Definition 3.1. The unique strong solution of SDE (3) is called the *square of the δ-dimensional Bessel process* started at Z_0.

The process $\rho_t = \sqrt{Z_t}$ is called the *δ-dimensional Bessel process* started at $\rho_0 = \sqrt{Z_0}$.

It is well-known that, for $\delta > 1$, the δ-dimensional Bessel process satisfies the SDE of the form:

$$X_t = X_0 + \int_0^t \frac{\delta - 1}{2X_s} I(X_s \neq 0)\,ds + B_t \tag{4}$$

(see [2; Ch. XI, §1]).

The following theorem is the main result of the present paper.

Theorem 3.2. (i) *The δ-dimensional $(\delta > 1)$ Bessel process is the unique (with fixed X_0 and B) nonnegative solution of (4). Moreover, it is a strong solution.*

(ii) *If $\delta \geq 2$ and $X_0 \neq 0$, then the pathwise uniqueness holds for (4).*

(iii) *If $1 < \delta < 2$ or $X_0 = 0$, then*

a) *there exist other strong solutions (with the same X_0 and B);*

b) *there exist weak solutions;*

c) *the uniqueness in law does not hold for (4).*

Proof. **(i)** Suppose that $(\widetilde{\rho}, B)$ is a nonnegative solution of (4). Then, by Itô's formula,

$$\widetilde{\rho}_t^2 = \widetilde{\rho}_0^2 + \int_0^t (\delta - 1)\, I(\widetilde{\rho}_s \neq 0)\,ds + 2\int_0^t \widetilde{\rho}_s\,dB_s + \int_0^t ds =$$

$$= \widetilde{\rho}_0^2 + \delta t - \int_0^t (\delta - 1) I(\widetilde{\rho}_s = 0)\,ds + 2\int_0^t |\widetilde{\rho}_s|\,dB_s.$$

Let $L_t^x(\widetilde{\rho})$ denote the local time of the process $\widetilde{\rho}$ at the point x. Then

$$\int_0^t I(\widetilde{\rho}_s = 0)\,ds = \int_0^t I(\widetilde{\rho}_s = 0)\,d\langle\widetilde{\rho}\rangle_s = \int_{\mathbb{R}} I(x = 0)\,L_t^x(\widetilde{\rho})\,dx = 0.$$

Thus, $(\widetilde{\rho}^2, B)$ is a solution of (3). The pair (ρ^2, B) is also a solution of (3) (here, ρ is the Bessel process started at X_0). Because of the pathwise uniqueness for (3), the processes $\widetilde{\rho}^2$ and ρ^2 are indistinguishable. As ρ and $\widetilde{\rho}$ are nonnegative, we deduce that ρ and $\widetilde{\rho}$ are indistinguishable.

The Bessel process is a strong solution of (4) since $\rho = \sqrt{\rho^2}$ and (ρ^2, B) is a strong solution of (3).

(ii) Suppose that $(\widetilde{\rho}, B)$ is a solution of (4) with $X_0 > 0$. Then

$$\widetilde{\rho}_t^2 = \widetilde{\rho}_0^2 + \delta t + 2\int_0^t \widetilde{\rho}_s\, dB_s = \widetilde{\rho}_0^2 + \delta t + 2\int_0^t |\rho_s|\, d\widetilde{B}_s,$$

where

$$\widetilde{B}_t = \int_0^t \operatorname{sgn} \widetilde{\rho}_s\, dB_s.$$

Thus, $(\widetilde{\rho}^2, \widetilde{B})$ is a solution of (3). By Proposition 1.5, the processes $\widetilde{\rho}^2$ and ρ^2 are equal in law. The Bessel processes have the following property (see [2; Ch. XI, §1]):

$$\delta \geq 2,\ \rho_0 > 0 \implies \mathsf{P}\bigl\{\forall t \geq 0,\ \rho_t > 0\bigr\} = 1.$$

Thus, $\mathsf{P}\{\forall t \geq 0, \widetilde{\rho}_t^2 > 0\} = 1$. This, together with the continuity of $\widetilde{\rho}$ and the condition $\widetilde{\rho}_0 > 0$, implies that $\mathsf{P}\{\forall t \geq 0, \widetilde{\rho}_t > 0\} = 1$. Therefore, $\widetilde{B} = B$ and $(\widetilde{\rho}^2, B)$ is a solution of (3). Due to the pathwise uniqueness for (3), the processes $\widetilde{\rho}^2$ and ρ^2 are indistinguishable. Finally, $\widetilde{\rho}$ and ρ are positive, and therefore, they are also indistinguishable.

(iii) a) Since the δ-dimensional Bessel process ρ is a strong solution of (4), there exists a measurable map $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ such that $\rho = \Phi(B)$ a.s. (see Theorem 2.1 (i)). We may write

$$\Phi_t(B(\omega)) = \Phi_0(B(\omega)) + \int_0^t \frac{\delta - 1}{2\Phi_s(B(\omega))}\, I\bigl(\Phi_s(B(\omega)) \neq 0\bigr)\, ds + B_t(\omega)$$

for P-a.e. ω.

Let us first consider the case $X_0 = 0$. Set $\widetilde{B}(\omega) = -B(\omega)$. Applying Theorem 2.1 (ii), we arrive at

$$-\Phi_t(-B(\omega)) = \int_0^t \frac{\delta - 1}{-2\Phi_s(-B(\omega))}\, I\bigl(-\Phi_s(-B(\omega)) \neq 0\bigr)\, ds + B_t(\omega).$$

This means that $(\widetilde{\rho}, B)$ is a strong solution of (4), where $\widetilde{\rho}(\omega) = -\Phi(-B(\omega))$. Moreover, $\rho_0 = \widetilde{\rho}_0 = 0$. Thus, ρ and $\widetilde{\rho}$ are two strong solutions of (4) with the same X_0 and B. They are different since ρ is positive while $\widetilde{\rho}$ is negative.

Now, suppose that $X_0 > 0$, $1 < \delta < 2$. Let ρ be the δ-dimensional Bessel process started at X_0. Consider the stopping time $\tau(\omega) = \inf\{t \geq 0 : \rho_t(\omega) = 0\}$. According to the properties of the Bessel processes (see [2; Ch. XI, §1]), we have $\mathsf{P}\{\tau < \infty\} = 1$. Set

$$\widetilde{B}_t = \int_0^t \bigl(I(s \leq \tau) - I(s > \tau)\bigr)\, dB_s,$$

$$\widetilde{\rho}_t(\omega) = \begin{cases} \Phi_t(B(\omega)) & \text{if } t \leq \tau(\omega), \\ -\Phi_t(\widetilde{B}(\omega)) & \text{if } t > \tau(\omega). \end{cases}$$

One can verify in the same way as above that (ρ, B) and $(\widetilde{\rho}, B)$ are two different strong solutions of (4) with $\rho_0 = \widetilde{\rho}_0$.

b) Let ξ be a Bernoulli random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ that is independent of B. Set

$$\bar{\rho}_t(\omega) = \begin{cases} \rho_t(\omega) & \text{if } \xi(\omega) = 1, \\ \widetilde{\rho}_t(\omega) & \text{if } \xi(\omega) = -1, \end{cases}$$

with ρ and $\widetilde{\rho}$ defined above. Then $(\bar{\rho}, B)$ is a solution of (4). It is a weak solution since ξ is measurable with respect to $\sigma(\bar{\rho}_s;\ s \geq 0)$ while ξ is independent of B.

c) This statement is obvious as the laws of ρ and $\widetilde{\rho}$ are not equal. □

`Remark`. SDE (4) with $1 < \delta < 2$, $X_0 > 0$ provides an example of an equation possessing non-Markov strong solutions. Indeed, let us consider the process

$$\widehat{\rho}_t(\omega) = \begin{cases} \rho_t(\omega) & \text{if } t \leq \tau(\omega), \\ \rho_t(\omega) & \text{if } t > \tau(\omega) \text{ and } \rho_{\tau/2}(\omega) > 1, \\ \widetilde{\rho}_t(\omega) & \text{if } t > \tau(\omega) \text{ and } \rho_{\tau/2}(\omega) \leq 1, \end{cases}$$

with ρ, $\widetilde{\rho}$ and τ defined above. It is easy to verify that $(\widehat{\rho}, B)$ is a non-Markov strong solution of (4). □

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