CAPITAL ALLOCATION AND RISK CONTRIBUTION WITH DISCRETE-TIME COHERENT RISK

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Abstract. We define the *capital allocation* and the *risk contribution* for discrete-time coherent risk measures and provide several equivalent representations of these objects. The formulations and the proofs are based on two instruments introduced in the paper: a probabilistic notion of the *extreme system* and a geometric notion of the *generator*. These notions are also of interest on their own and are important for other applications of coherent risk measures. All the concepts and results are illustrated by the JP Morgan's Risk Metrics model.

Key words and phrases. Capital allocation, determining system, dynamic coherent risk measure, dynamic Weighted V@R, extreme system, generator, risk contribution, Risk Metrics.

1 Introduction

Overview. Artzner, Delbaen, Eber, and Heath [1], [2] introduced the concept of a coherent risk measure. These risk measures are static in the sense that they take into account only the situation at the terminal date. But it is also important to consider their dynamic extensions, which take into account the information structure, i.e. the filtration. There are at least 3 reasons:

- (Mathematical argument) One can apply coherent risk measures to problems like pricing, hedging, and optimization. For dynamic models, one can apply both static and dynamic risk measures (a static risk measure is simply applied to the terminal wealth of a portfolio). A big advantage of the latter class is that it allows one to use a powerful tool of the backward induction.
- (Financial argument) Static risk measures do not take into account the timing of payments in a stream, while this timing is important from the viewpoint of the funding liquidity risk.
- (Ideological argument) If one measures risk through static risk measures, then a problem immediately arises: what risk measures should be taken for various time horizons? For example, if one employs the family of Tail V@Rs indexed by $\lambda \in (0, 1]$, then what λ should be used for one day, one month, and one year so that risk measures employed for various horizons are consistent in a certain sense? In our opinion, this problem has no satisfactory solution. On the other hand, for dynamic risk measures, the time consistency is the basic property.

Dynamic risk measures were studied in a number of papers. However, in contrast to the static case, in the dynamic case there is still no unanimity about what the "right" risk measure should look like. There are 4 approaches:

- A risk measure is a map from random variables to real numbers. A random variable means the terminal capital of some portfolio, while the number means the risk of the portfolio at the initial time. In this case a dynamic risk measure coincides with a static risk measure.
- A risk measure is a map from random processes to real numbers. A process means the capital process of some portfolio, while the number means the risk of the portfolio at the initial time. This approach was taken in the papers by Artzner et al. [3], Cheridito et al. [4], [5], Frittelli and Scandolo [21].
- A risk measure is a map from random variables to random processes. A random variable means the terminal capital of some portfolio, while the value of the process at time n means the risk of the portfolio given the information available at time n. This approach was taken in the papers by Detlefsen and Scandolo [14], Föllmer and Penner [17], Frittelli and Rosazza Gianin [20], Roorda et al. [27], and Weber [30].
- A risk measure is a map from random processes to random processes. The first process means the capital process of some portfolio, while the value of the second process at time n means the risk of the portfolio given the information available at time n. This approach was taken in the papers by Cheridito et al. [6], Cheridito and Kupper [7], Jobert and Rogers [23], and Riedel [26].

Goal of the paper. We take as the basis the representation of a discrete-time coherent risk measure from Cheridito and Kupper [7] with an extension to unbounded processes. This extension is necessary because almost all distributions used in concrete models are unbounded. In particular, this is the case for the JP Morgan's Risk Metrics model, on which we illustrate our concepts and results.

We give the definition and provide two representations of the *capital allocation*. We also give the definition and provide three representations of the *risk contribution*. For both the capital allocation and the risk contribution, we provide a geometric representation based on the notion of a *generator* and a probabilistic representation based on the notion of an *extreme system* (the former concept is also the core of the proof of the third, analytic, representation of the risk contribution). These two notions are introduced in this paper, while their static counterparts were introduced in [8]. The results of [8] and [9] show that these notions are very convenient in applications of coherent risk measures to various problems of finance.

Let us remark that capital allocation and/or risk contribution for coherent risk measures was considered by Cherny [8], Delbaen [12], Denault [13], Fischer [15], Kalkbrenner [24], Overbeck [25], and Tasche [28], but all these papers deal with the static risk measures. Our results on the representation of capital allocation and risk contribution might be viewed as the dynamic counterpart of the results from [8].

Structure of the paper. In Section 2, we extend dynamic risk measures to unbounded processes. We also introduce and study an important class of risk measures — the dynamic Weighted V@R.

In Section 3, we introduce the notion of a generator. The main result of the section is Theorem 3.2, which establishes the relation between risks and generators.

In Section 4, we introduce the notion of an extreme system. The main result of the section is Theorem 4.2, which provides the form of the generator for the extreme system.

In Section 5, we give the definition of the capital allocation. The main result of the section is Theorem 5.2, which provides two representations of the capital allocation: the geometric one is based on generators and the probabilistic one is based on extreme systems.

In Section 6, we give the definition of the risk contribution. The main result of the section is Theorem 6.2, which provides three representations of the risk contribution: the geometric one is based on generators, the probabilistic one is based on extreme systems, and the analytic one is based on marginal risks.[1]

To illustrate our results, we find the generator, the extreme system, the capital allocation, and the risk contribution for the JP Morgan's Risk Metrics model combined with the dynamic Weighted V@R risk measure.

Some technical statements are gathered in the Appendix.

2 Discrete-Time Coherent Risk

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P})$ be a discrete-time filtered probability space. Let $\mathcal{D} = (\mathcal{D}_n)_{n=1,\dots,N}$ be a system of sets of random variables with the properties:

- any random variable Z from $\mathcal{D}_n$ is positive, $\mathcal{F}_n$-measurable, and satisfies the inequality $\mathsf{E}(Z \mid \mathcal{F}_{n-1}) \le 1$;
- $\mathcal{D}_n$ is non-empty, L^1-closed, uniformly integrable, and $\mathcal{F}_{n-1}$*-convex*, i.e. for any $Z_1, Z_2 \in \mathcal{D}_n$ and any $[0,1]$-valued $\mathcal{F}_{n-1}$-measurable random variable λ, we have $\lambda Z_1 + (1-\lambda) Z_2 \in \mathcal{D}_n$.

Definition 2.1. Let X be a one-dimensional $(\mathcal{F}_n)$-adapted process. The *coherent utility* of X is the $[-\infty, \infty]$-valued process $u(X) = (u_n(X))_{n=0,\dots,N}$ defined as: $u_N(X) = 0$,

$$u_{n-1}(X) = \operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(X_n + u_n(X)) \mid \mathcal{F}_{n-1}), \quad n = N, \dots, 1,$$

where $\mathsf{E}(\xi \mid \mathcal{G})$ is understood as $\mathsf{E}(\xi^+ \mid \mathcal{G}) - \mathsf{E}(\xi^- \mid \mathcal{G})$ with the convention $\infty - \infty = -\infty$. The corresponding *coherent risk* is $\rho(X) = -u(X)$. The system $\mathcal{D}$ will be called the *determining system* of u (or ρ).

`Remarks`. (i) From the financial point of view, X describes a stream of cash flows, i.e. X_n is the cash flow at time n, while $\rho_n(X)$ means the risk of the residual part of the stream after time n. Of course, equivalently one can use cumulative cash flow streams and cumulative risks (and this is done in most papers). We follow the first approach because this enables one to avoid the dependence of risk on the initial date 0. Moreover, the use of payment streams and residual risks leads to nicer formulations of the pricing and hedging theorems in the forthcoming paper.

(ii) Our conditions on $\mathcal{D}_n$ are essential for most of the results below. Moreover, they are automatically satisfied for natural dynamic risk measures like Weighted V@R (see Example 2.4).

(iii) Let $(\mathcal{D}'_n)_{n=1,\dots,N}$ be another system such that $\mathcal{D}'_n$ consists of integrable random variables and suppose that this system defines the same coherent utility u. Then $\mathcal{D}'_n \subseteq \mathcal{D}_n$ for any n, which justifies the term "determining system" we are using. Indeed, suppose that $\mathcal{D}'_m \not\subseteq \mathcal{D}_m$ for some m. By the Hahn-Banach theorem, there exists $\xi \in L^\infty(\mathcal{F}_m)$

[1] In fact, Theorems 5.2 and 6.2 are easy consequences of Theorems 3.2 and 4.2.

such that $\inf_{Z\in\mathcal{D}'_m} \mathsf{E}Z\xi < \inf_{Z\in\mathcal{D}_m} \mathsf{E}Z\xi$. Take $X_n = \xi I(n = m)$, $n = 0,\dots,N$. Then $u_n(X) = 0$ for $n \ge m$. Furthermore, by Lemma 2.3, there exists $Z_* \in \mathcal{D}_m$ such that $u_{m-1}(X) = \mathsf{E}(Z_*\xi\,|\,\mathcal{F}_{m-1})$. We can find $Z \in \mathcal{D}'_m$ such that $\mathsf{E}Z\xi < \mathsf{E}Z_*\xi$. Thus, $\operatorname{essinf}_{Z\in\mathcal{D}'_m} \mathsf{E}(Z\xi\,|\,\mathcal{F}_{m-1}) \ne u_{m-1}(X)$, which is a contradiction.

(iv) Föllmer and Schied [18] and Frittelli and Rosazza Gianin [19] introduced (in the static setting) the notion of a convex risk measure, which is more general than the notion of a coherent risk measure. In fact, most of the papers on dynamic risks mentioned above deal with convex rather than coherent risks. However, many notions below (for example, the notion of a generator) make sense only for coherent risks. More important, in applications to pricing, hedging, and optimality, coherent risk measures lead to more explicit results. For this reason, we restrict our attention to coherent risks. □

We will say that $\mathcal{D}$ is *probabilistic* if $\mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) = 1$ for any n and any $Z \in \mathcal{D}_n$. This property is very natural and, in fact, many papers on dynamic risks automatically assume it. However, following Cheridito et al. [6], Cheridito and Kupper [7], and Jobert and Rogers [23], we allow for the inequality $\mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) \le 1$. This accounts for the effect, which might be called the discounting of risk. To illustrate it, consider a simple example: $\mathcal{D}_n = \{a\}$, where $a \in [0,1]$. If we consider the process $X_n = I(n = N)$ (corresponding to the portfolio obtaining 1 unit of money at time N), then $u_0(X) = a^N$. Now, the choice of a is linked to the question: is it true that 1 unit of (discounted) money obtained at time N has the same utility as 1 unit of money obtained at time 0? If the time horizon N is 100 days, then the common sense suggests a positive answer; however, if N is 100 years, then the common sense suggests a negative answer. Thus, for short time periods one can restrict attention to probabilistic risk measures. However, for long time periods it is reasonable to consider more general risk measures with $\mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) \le 1$ for $Z \in \mathcal{D}_n$.

`Remark`. If there exists $a \in (0,1]$ such that $\mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) = a$ for any n and any $Z \in \mathcal{D}_n$ (this is the case for our basic example — Weighted V@R; see Example 2.4), then the risk measure is reduced to a probabilistic one. Namely, by the backward induction one easily checks that, for any X and any n, $u_n(X) = a^{-n}u'_n(X')$, where $X'_n = a^n X_n$ and u' is the coherent utility with the determining system $\mathcal{D}'_n = a^{-1}\mathcal{D}_n$, $n = 1,\dots,N$. □

The result below establishes the relationship between our definition of a coherent risk measure and the representations from Cheridito and Kupper [7] and Riedel [26]. Let us introduce the notation

$$\overline{\mathcal{D}} = \Bigl\{\prod_{n=1}^N Z_n : Z_n \in \mathcal{D}_n \cup \{1\}\Bigr\}, \qquad \widetilde{\mathcal{D}} = \Bigl\{\prod_{n=1}^N Z_n : Z_n \in \mathcal{D}_n\Bigr\}.$$

For a set $\mathcal{E}$ of random variables, we define the space

$$L^1(\mathcal{E}) = \bigl\{X \in L^0 : \lim_{a\to\infty} \sup_{Z\in\mathcal{E}} \mathsf{E}Z|X|I(|X| > a) = 0\bigr\},$$

where L^0 denotes the space of all random variables. It is easy to check that $L^1(\mathcal{E})$ is a linear space. Throughout the paper, we identify probability measures that are absolutely continuous with respect to P with their densities with respect to P; conditional expectations $\mathsf{E}_{\mathsf{Q}}(\xi\,|\,\mathcal{F}_n)$ are understood with the convention $\mathsf{E}_{\mathsf{Q}}(\xi\,|\,\mathcal{F}_n) = 0$ on the set $\bigl\{\frac{d\mathsf{Q}|\mathcal{F}_n}{d\mathsf{P}|\mathcal{F}_n} = 0\bigr\}$. We will use the notation

$$\operatorname*{argessmin}_{\xi\in A} \xi = \bigl\{\xi \in A : \xi = \operatorname*{essinf}_{\xi'\in A} \xi'\bigr\}.$$

Proposition 2.2. *If $X_n \in L^1(\overline{\mathcal{D}})$ for any n, then there exist $Z_n^* \in \mathcal{D}_n$ such that*

$$u_n(X) = \operatorname*{essinf}_{Z_k \in \mathcal{D}_k} \mathsf{E}\biggl(\sum_{k=n+1}^{N} Z_{n+1}...Z_k X_k \bigg| \mathcal{F}_n\biggr) = \mathsf{E}\biggl(\sum_{k=n+1}^{N} Z_{n+1}^*...Z_k^* X_k \bigg| \mathcal{F}_n\biggr), \quad n = 0, ..., N.$$

If moreover $\mathcal{D}$ is probabilistic, then there exists $\mathsf{Q}_* \in \widetilde{\mathcal{D}}$ such that

$$u_n(X) = \operatorname*{essinf}_{\mathsf{Q} \in \widetilde{\mathcal{D}}} \mathsf{E}_{\mathsf{Q}}\biggl(\sum_{k=n+1}^{N} X_k \bigg| \mathcal{F}_n\biggr) = \mathsf{E}_{\mathsf{Q}_*}\biggl(\sum_{k=n+1}^{N} X_k \bigg| \mathcal{F}_n\biggr), \quad n = 0, \dots, N.$$

`Proof`. We should check only the first statement. Let us prove the following property going backwards from $n = N$ to $n = 0$: $u_n(X) \in L^1(\overline{\mathcal{D}})$ and there exist $Z_{n+1}^* \in \mathcal{D}_{n+1}, \dots, Z_N^* \in \mathcal{D}_N$ such that

$$u_m(X) = \mathsf{E}\biggl(\sum_{k=m+1}^{N} Z_{m+1}^* \dots Z_k^* X_k \bigg| \mathcal{F}_m\biggr), \quad m = n, \dots, N. \tag{2.1}$$

Suppose that this statement is true for n and let us prove it for $n-1$. Denote $\xi = X_n + u_n(X)$. According to Lemma 2.3, there exists $Z_n^* \in \operatorname{argessmin}_{Z \in \mathcal{D}_n} \mathsf{E}(Z\xi \,|\, \mathcal{F}_{n-1})$. Fix $\varepsilon > 0$. We can find $a \in \mathbb{R}_+$ such that, for $\xi_1 = \xi I(|\xi| \le a)$ and $\xi_2 = \xi I(|\xi| > a)$, we have $\mathsf{E}Z|\xi_2| < \varepsilon$ for any $Z \in \overline{\mathcal{D}}$. Then $u_{n-1}(X) = \eta_1 + \eta_2$, where $\eta_i = \mathsf{E}(Z_n^* \xi_i \,|\, \mathcal{F}_{n-1})$. Clearly, $|\eta_1| \le a$, while for any $Z = Z_1 \dots Z_N \in \overline{\mathcal{D}}$, we have

$$\mathsf{E}Z|\eta_2| = \mathsf{E}Z_1 \dots Z_{n-1}|\eta_2| \le \mathsf{E}(Z_1 \dots Z_{n-1} Z_n^* |\xi_2| \,|\, \mathcal{F}_{n-1}) = \mathsf{E}Z_1 \dots Z_{n-1} Z_n^* |\xi_2| < \varepsilon.$$

Thus, $u_{n-1}(X) \in L^1(\overline{\mathcal{D}})$. Equality (2.1) for $n-1$ is obvious.

By the backward induction, one easily checks that, for any $Z_1 \in \mathcal{D}_1, \dots, Z_N \in \mathcal{D}_N$,

$$u_n(X) \le \mathsf{E}\biggl(\sum_{k=n+1}^{N} Z_{n+1}...Z_k X_k \bigg| \mathcal{F}_n\biggr), \quad n = 0, \dots, N.$$

`Remarks`. (i) If $\mathcal{D}$ is probabilistic, then $u_n(X)$ depends only the the cumulative cash flow remaining after time n and does not depend on the timing of payments (provided that $X_n \in L^1(\overline{\mathcal{D}})$ for any n).

(ii) Without the assumption $X_n \in L^1(\overline{\mathcal{D}})$, the above statement is false. As an example, let $N = 2$, $\mathcal{F}_0$ be trivial, $\mathcal{D}_n = \{1\}$, and ξ be a non-integrable $\mathcal{F}_2$-measurable random variable with $\mathsf{E}(\xi \,|\, \mathcal{F}_1) = 0$. Then, for $X_n = \xi I(n = 2)$, we have $u_0(X) = 0$, while $\inf_{\mathsf{Q} \in \widetilde{\mathcal{D}}} \mathsf{E}_{\mathsf{Q}}(X_1 + X_2) = -\infty$. □

Lemma 2.3. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, $\mathcal{E}$ be a non-empty $\mathcal{G}$-convex L^1-closed uniformly integrable set, and $X \in L^1(\mathcal{E})$. Then* $\operatorname{argessmin}_{Z \in \mathcal{E}} \mathsf{E}(ZX \,|\, \mathcal{G}) \neq \emptyset$.

`Proof`. Find a sequence $Z_n \in \mathcal{E}$ such that $\mathsf{E}Z_n X \to \inf_{Z \in \mathcal{E}} \mathsf{E}ZX$. By the Dunford-Pettis criterion, $\mathcal{E}$ is relatively weakly compact. Being L^1-closed and convex, it is weakly closed by the Hahn-Banach theorem. Thus, $\mathcal{E}$ is weakly compact. Consequently, (Z_n) has a weak limit point $Z_* \in \mathcal{E}$. As the map $\mathcal{E} \ni Z \mapsto \mathsf{E}ZX$ is weakly continuous, we have $\mathsf{E}Z_* X = \inf_{Z \in \mathcal{E}} \mathsf{E}ZX$. Take $Z \in \mathcal{E}$ and consider the set $A = \{\mathsf{E}(ZX \,|\, \mathcal{G}) < \mathsf{E}(Z_* X \,|\, \mathcal{G})\}$. Then $\widetilde{Z} := I_A Z + I_{A^c} Z_* \in \mathcal{E}$ and the inequality $\mathsf{E}ZX \ge \mathsf{E}Z_* X$ shows that $\mathsf{P}(A) = 0$. Thus, $Z_* \in \operatorname{argessmin}_{Z \in \mathcal{E}} \mathsf{E}(ZX \,|\, \mathcal{G})$. □

We now turn to our basic example of a coherent risk measure.

Example 2.4 (Weighted V@R). **(i)** In the static case one of the basic examples of coherent risk is *Tail V@R* defined as $\rho_\lambda = -u_\lambda$,

$$u_\lambda(X) = \inf_{Z\in\mathcal{D}_\lambda} \mathsf{E}ZX, \quad X \in L^0,$$

where $\lambda \in (0,1]$ and $\mathcal{D}_\lambda = \{Z : 0 \le Z \le \lambda^{-1}, \mathsf{E}Z = 1\}$. The expectation $\mathsf{E}_\mathsf{Q}X$ is understood as $\mathsf{E}_\mathsf{Q}X^+ - \mathsf{E}_\mathsf{Q}X^-$ with the convention $\infty - \infty = -\infty$.

Its natural discrete-time analog is provided by taking

$$\mathcal{D}_n = \{Z : Z \text{ is } \mathcal{F}_n\text{-measurable}, 0 \le Z \le \lambda^{-1}, \text{ and } \mathsf{E}(Z|\mathcal{F}_{n-1}) = 1\}, \quad n = 1, \dots, N.$$

(ii) A more general class of static coherent risks is provided by *Weighted V@R* defined as $\rho_\mu = -u_\mu$,

$$u_\mu(X) = \int_{(0,1]} \rho_\lambda(X)\mu(d\lambda), \quad X \in L^0, \tag{2.2}$$

where μ is a probability measure on $(0,1]$. The integral $\int_{(0,1]} f(x)\mu(dx)$ is understood as $\int_{(0,1]} f^+(x)\mu(dx) - \int_{(0,1]} f^-(x)\mu(dx)$ with the convention $\infty - \infty = -\infty$. This functional can be represented as

$$u_\mu(X) = \inf_{Z\in\mathcal{D}_\mu} \mathsf{E}ZX, \quad X \in L^0, \tag{2.3}$$

where

$$\mathcal{D}_\mu = \{Z : Z \ge 0, \mathsf{E}Z = 1, \mathsf{E}(Z-x)^+ \le \Phi_\mu(x) \ \forall x \in \mathbb{R}_+\} \tag{2.4}$$

and

$$\Phi_\mu(x) = \sup_{y\in[0,1]} \left[\int_0^y \int_{[z,1]} \lambda^{-1}\mu(d\lambda)dz - xy \right], \quad x \in \mathbb{R}_+ \tag{2.5}$$

(see [10; Th. 4.6]). The class Weighted V@R is analytically convenient and includes as particular cases several very important coherent risks. In particular, if $\mu(dx) = \mathrm{B}(2, \alpha-1)^{-1}x(1-x)^{\alpha-2}dx$ with $\alpha \in \mathbb{N}$, then

$$u_\mu(X) = \mathsf{E}\min\{X_1, \dots, X_\alpha\},$$

where $X_1, \dots, X_\alpha$ are independent copies of X. This risk measure was introduced in [11], and we call it *Extreme V@R* or *XV@R* for short.

The above representation enables us to extend Weighted V@R to the dynamic case by setting

$$\begin{aligned} \mathcal{D}_n = \{Z : Z \text{ is } \mathcal{F}_n\text{-measurable}, Z \ge 0, \mathsf{E}(Z|\mathcal{F}_{n-1}) = 1, \\ \text{and } \mathsf{E}((Z-x)^+|\mathcal{F}_{n-1}) \le \Phi_\mu(x) \ \forall x \in \mathbb{R}_+\}, \quad n = 1, \dots, N. \end{aligned} \tag{2.6}$$

The sets $\mathcal{D}_n$ satisfy the conditions imposed at the beginning of this section (see Lemma 2.5).

Let us remark that the functional ρ_μ can be defined by (2.2) also for a positive (not necessarily probabilistic) finite measure μ on $(0,1]$ and representation (2.3)–(2.5) remains valid with the condition $\mathsf{E}Z = 1$ replaced by $\mathsf{E}Z = \mu((0,1])$. This enables us to extend the dynamic Weighted V@R to measures μ with $\mu((0,1] \le 1$ simply by replacing in (2.6) the condition $\mathsf{E}(Z|\mathcal{F}_{n-1}) = 1$ by the condition $\mathsf{E}(Z|\mathcal{F}_{n-1}) = \mu((0,1])$. □

The lemma below provides a representation of the dynamic Weighted V@R. It is known that $u_\mu(X)$ depends only on the distribution of X, so that there exists a functional $\widetilde{u}_\mu$ defined on distributions such that $u_\mu(X) = \widetilde{u}_\mu(\mathsf{Law}\, X)$, $X \in L^0$.

Lemma 2.5. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, and μ be a positive measure on $(0,1]$. Define*

$$\mathcal{D} = \{Z : Z \text{ is } \mathcal{F}\text{-measurable, } Z \geq 0,\ \mathsf{E}(Z\,|\,\mathcal{G}) = \mu((0,1]), \\ \text{and } \mathsf{E}((Z-x)^+\,|\,\mathcal{G}) \leq \Phi_\mu(x)\ \forall x \in \mathbb{R}_+\}.$$

Then $\mathcal{D}$ is $\mathcal{G}$-convex, L^1-closed, uniformly integrable, and

$$\operatorname*{essinf}_{Z\in\mathcal{D}} \mathsf{E}(ZX\,|\,\mathcal{G}) = \widetilde{u}_\mu(\mathsf{Law}(X\,|\,\mathcal{G})), \quad X \in L^0. \tag{2.7}$$

`Proof`. For $Z_1, Z_2 \in \mathcal{D}$ and $[0,1]$-valued $\mathcal{G}$-measurable λ, we have

$$\mathsf{E}((\lambda Z_1 + (1-\lambda)Z_2 - x)^+\,|\,\mathcal{G}) \leq \lambda\mathsf{E}((Z_1-x)^+\,|\,\mathcal{G}) + (1-\lambda)\mathsf{E}((Z_2-x)^+\,|\,\mathcal{G}) \leq \Phi_\mu(x), \quad x \in \mathbb{R}_+,$$

which proves the $\mathcal{G}$-convexity.

The L^1-closedness is clear, while the uniform integrability follows from the estimate $\mathsf{E}(Z-x)^+ \leq \Phi_\mu(x)$ and the property $\lim_{x\to\infty}\Phi_\mu(x) = 0$.

Let us prove (2.7). Fix $X \in L^0$. Set $\mathsf{Q}_\omega = \mathsf{Law}(X\,|\,\mathcal{G})(\omega)$, $F_\omega(x) = \mathsf{Q}_\omega((-\infty, x])$, $q_\lambda(\omega) = \inf\{x : F_\omega(x) > \lambda\}$, and

$$r_\lambda(\omega) = \frac{1 - \lambda^{-1}F_\omega(q_\lambda(\omega)-)}{\Delta F_\omega(q_\lambda(\omega))}\, I(\Delta F_\omega(q_\lambda(\omega)) > 0),$$

where $\lambda \in (0,1)$ and $\Delta F(x) = F(x) - F(x-)$. The equality

$$\{q_\lambda < \alpha\} = \{F(\alpha-) > \lambda\}, \quad \alpha \in \mathbb{R}$$

shows that q_λ is $\mathcal{G}$-measurable. The equality

$$\{\Delta F(q_\lambda) \geq \alpha\} = \bigcap_{n=1}^{\infty} \bigcup_{\substack{a_1<a_2\in\mathbb{Q}_+ \\ a_2-a_1<n^{-1}}} \{F(a_1) \leq \lambda\} \cap \{F(a_2) > \lambda\} \cap \{F(a_2) - F(a_1) \geq \alpha\}, \quad \alpha \in (0,1]$$

shows that r_λ is $\mathcal{G}$-measurable. Set

$$Z_\lambda = \lambda^{-1}I(X < q_\lambda) + r_\lambda I(X = q_\lambda).$$

By the standard monotone class arguments, for any positive measurable function f,

$$\mathsf{E}(f(X, q_\lambda, r_\lambda)\,|\,\mathcal{G})(\omega) = \int_{\mathbb{R}} f(x, q_\lambda(\omega), r_\lambda(\omega))\mathsf{Q}_\omega(dx) \quad \text{a.s.}$$

It follows that

$$\mathsf{E}(Z_\lambda X\,|\,\mathcal{G})(\omega) = \int_{\mathbb{R}} x[\lambda^{-1}I(x < q_\lambda(\omega)) + r_\lambda(\omega)I(x = q_\lambda(\omega))]\mathsf{Q}_\omega(dx) = \widetilde{u}_\lambda(\mathsf{Q}_\omega) \quad \text{a.s.,}$$

where $\widetilde{u}_\lambda = \widetilde{u}_{\delta_\lambda}$ (the last equality follows from a well-known representation of $\widetilde{u}_\lambda$; see [10; Prop. 2.7]).

Let

$$a_k^n = 1 - \sum_{i=0}^{k-1} \frac{1}{2^{[i/n]+1}n}, \quad \alpha_k^n = \mu((a_{k+1}^n, a_k^n]), \quad n \in \mathbb{N},\ k \in \mathbb{Z}_+.$$

Consider the measures $\mu^n = \sum_{k=0}^{\infty} \alpha_k^n \delta_{a_k^n}$ and the random variables $Z^n = \sum_{k=0}^{\infty} \alpha_k^n Z_{a_k^n}$. Let us check that $Z^n \in \mathcal{D}$. We have $\mathsf{E}(Z_\lambda | \mathcal{G}) = 1$, so that $\mathsf{E}(Z^n | \mathcal{G}) = \mu((0,1])$. Each μ^n stochastically dominates μ, so that $\Phi_{\mu^n} \le \Phi_\mu$. Denote

$$\Psi_{\mu^n}(x) = \int_0^x \int_{[y,1]} \lambda^{-1} \mu^n(d\lambda), \quad n \in \mathbb{N},\ x \in [0,1].$$

Then

$$(\Psi_{\mu^n})'_+(a_k^n) = \int_{(a_k^n,1]} \lambda^{-1} \mu^n(d\lambda) = \sum_{i=0}^{k-1} \frac{\alpha_i^n}{a_i^n}, \quad n \in \mathbb{N},\ k \in \mathbb{Z}_+,$$

$$\Psi_{\mu^n}(a_k^n) = a_k^n \sum_{i=0}^{k-1} \frac{\alpha_i^n}{a_i^n} + \sum_{i=k}^{\infty} a_i^k, \quad n \in \mathbb{N},\ k \in \mathbb{Z}_+.$$

Consequently,

$$\Phi_{\mu^n}\Bigl(\sum_{i=0}^{k-1} \frac{\alpha_i^n}{a_i^n}\Bigr) = \Psi_{\mu^n}(a_k^n) - a_k^n \sum_{i=0}^{k-1} \frac{\alpha_i^n}{a_i^n} = \sum_{i=k}^{\infty} a_i^n, \quad n \in \mathbb{N},\ k \in \mathbb{Z}_+.$$

Furthermore, $Z_\lambda \le \lambda^{-1}$, so that

$$\mathsf{E}\biggl(\Bigl(Z^n - \sum_{i=0}^{k-1} \frac{\alpha_i^n}{a_i^n}\Bigr)^+ \bigg| \mathcal{G}\biggr) \le \sum_{i=k}^{\infty} a_i^n = \Phi_{\mu^n}\Bigl(\sum_{i=0}^{k-1} \frac{\alpha_i^n}{a_i^n}\Bigr), \quad n \in \mathbb{N},\ k \in \mathbb{Z}_+.$$

The function Φ_{μ^n} is linear between points of the form $\sum_{i=0}^{k-1} \alpha_i^n / a_i^n$, and the function $\mathsf{E}((Z^n - x)^+ | \mathcal{G})(\omega)$ is convex for a.e. ω. Therefore,

$$\mathsf{E}((Z^n - x)^+ | \mathcal{G}) \le \Phi_{\mu^n}(x) \le \Phi_\mu(x), \quad x \in \mathbb{R}_+.$$

Thus, $Z^n \in \mathcal{D}$.

We have

$$\mathsf{E}(Z^n X | \mathcal{G})(\omega) = \sum_{k=0}^{\infty} \alpha_k^n \widetilde{u}_{a_k^n}(\mathsf{Q}_\omega) = \widetilde{u}_{\mu^n}(\mathsf{Q}_\omega), \quad n \in \mathbb{N},$$

where the sum is understood as $-\infty$ if any of the summands equals $-\infty$. It is not hard to see that the right-hand side of this equality converges to $\widetilde{u}_\mu(\mathsf{Q}_\omega)$. As a result, $\widetilde{u}_\mu(\mathsf{Law}(X | \mathcal{G}))$ is $\mathcal{G}$-measurable and

$$\operatorname*{essinf}_{Z \in \mathcal{D}} \mathsf{E}(ZX | \mathcal{G}) \le \widetilde{u}_\mu(\mathsf{Law}(X | \mathcal{G})), \quad X \in L^0.$$

Let us prove the reverse inequality. Take $X \in L^0$, $Z \in \mathcal{D}$. Set $\mathsf{Q}_\omega = \mathsf{Law}(X, Z | \mathcal{G})(\omega)$ and let Q_ω^1 and Q_ω^2 denote its projections on the first and the second axis, respectively. We have

$$\mathsf{P}\Bigl(\int_{\mathbb{R}_+} (y - x)^+ \mathsf{Q}_\omega^2(dx) \le \Phi_\mu(x)\ \forall x \in \mathbb{Q}_+\Bigr) = 1.$$

Consequently, $\mathsf{Q}_\omega^2 \in \widetilde{\mathcal{D}}_\mu$ for a.e. ω, where

$$\widetilde{\mathcal{D}}_\mu = \Bigl\{\mathsf{Q} : \mathsf{Q} \text{ is a probability measure on } \mathbb{R}_+, \int_{\mathbb{R}_+} y \mathsf{Q}(dy) = \mu((0,1]), \text{ and } \int_{\mathbb{R}_+} (y - x)^+ \mathsf{Q}(dy) \le \Phi_\mu(x)\ \forall x \in \mathbb{R}_+\Bigr\}. \tag{2.8}$$

Now, it follows from [10; Th. 4.6] that

$$\mathsf{E}(ZX\,|\,\mathcal{G})(\omega) = \int_{\mathbb{R}^2} x^2x^1\mathsf{Q}_\omega(dx^1,dx^2) \geq \widetilde{u}_\mu(\mathsf{Q}^1_\omega) \quad \text{a.s.},$$

which proves the inequality

$$\operatorname*{essinf}_{Z\in\mathcal{D}} \mathsf{E}(ZX\,|\,\mathcal{G}) \geq \widetilde{u}_\mu(\mathsf{Law}(X\,|\,\mathcal{G})), \quad X \in L^0.$$

3 Generators

For a sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ and a measurable map C from Ω to the set $\mathcal{C}$ of non-empty convex compacts in $\mathbb{R}^d$, we denote by $L^0(\mathcal{G},C)$ the set of $\mathcal{G}$-measurable $\mathbb{R}^d$-valued random vectors Y such that $Y\in C$ a.s.

Definition 3.1. Let X be a d-dimensional $(\mathcal{F}_n)$-adapted process. The *generator* of X is the $\mathcal{C}$-valued process $G(X)=(G_n(X))_{n=0,\dots,N}$ defined as: $G_N(X)=\{0\}$,

$$G_{n-1}(X) = \operatorname*{essconv}_{Z\in\mathcal{D}_n,\,Y\in L^0(\mathcal{F}_n,G_n(X))} \mathsf{E}(Z(X_n+Y)\,|\,\mathcal{F}_{n-1}), \quad n=N,\dots,1,$$

where essconv denotes the essential closed convex hull (see Appendix).

The generator exists if and only if

$$\operatorname*{esssup}_{Z\in\mathcal{D}_n,\,Y\in L^0(\mathcal{F}_n,G_n(X))} \mathsf{E}(Z\|Y-X_{n-1}\|\,|\,\mathcal{F}_{n-1}) < \infty, \quad n=1,\dots,N.$$

This is equivalent to the finiteness of the random variables $u_n(X^i)$, $u_n(-X^i)$ for any n,i.

The following statement establishes the relationship between risks and generators. By φ_h we denote the support function

$$\varphi_h(C) = \min_{x\in C}\langle h,x\rangle, \quad C\in\mathcal{C},\ h\in\mathbb{R}^d.$$

Theorem 3.2. *If $G(X)$ exists, then*

$$u_n(\langle h,X\rangle) = \varphi_h(G_n(X)), \quad h\in\mathbb{R}^d,\ n=0,\dots,N.$$

`Proof`. We will prove this statement by the induction in n going backwards from N to 0. Suppose it is true for n and let us prove it for $n-1$. By Lemma A.1, the random set $G' = \operatorname{argmin}_{x\in G_n(X)}\langle h,x\rangle$ is $\mathcal{F}_n$-measurable. By Lemma A.2, there exists $Y_*\in L^0(\mathcal{F}_n,G')$. Then

$$u_n(\langle h,X\rangle) = \varphi_h(G_n(X)) = \langle h,Y_*\rangle = \operatorname*{essinf}_{Y\in L^0(\mathcal{F}_n,G_n(X))}\langle h,Y\rangle,$$

so that

$$\begin{aligned} u_{n-1}(\langle h,X\rangle) &= \operatorname*{essinf}_{Z\in\mathcal{D}_n} \mathsf{E}(Z(\langle h,X_n\rangle+\langle h,Y_*\rangle)\,|\,\mathcal{F}_{n-1}) \\ &= \operatorname*{essinf}_{Z\in\mathcal{D}_n,\,Y\in L^0(\mathcal{F}_n,G_n(X))} \mathsf{E}(Z(\langle h,X_n\rangle+\langle h,Y\rangle)\,|\,\mathcal{F}_{n-1}) \\ &= \operatorname*{essinf}_{Z\in\mathcal{D}_n,\,Y\in L^0(\mathcal{F}_n,G_n(X))} \langle h,\mathsf{E}(Z(X_n+Y)\,|\,\mathcal{F}_{n-1})\rangle \\ &= \varphi_h(G_{n-1}(X)), \end{aligned}$$

where the last equality follows from (a.2). □

Corollary 3.3. *If $\|X_n\| \in L^1(\overline{\mathcal{D}})$ for any n, then*

$$G_n(X) = \operatorname*{essconv}_{Z_k \in \mathcal{D}_k} \mathsf{E}\Big(\sum_{k=n+1}^{N} Z_{n+1} \dots Z_k X_k \,\Big|\, \mathcal{F}_n\Big), \quad n = 0, \dots, N.$$

If moreover $\mathcal{D}$ is probabilistic, then

$$G_n(X) = \operatorname*{essconv}_{\mathsf{Q} \in \widetilde{\mathcal{D}}} \mathsf{E}_{\mathsf{Q}}\Big(\sum_{k=n+1}^{N} X_n \,\Big|\, \mathcal{F}_n\Big), \quad n = 0, \dots, N.$$

`Proof`. We should check only the first equality. Denote its right-hand side by G'_n. Fix n. Applying Lemma A.4, Proposition 2.2, and Theorem 3.2, we can write

$$\varphi_h(G'_n) = \operatorname*{essinf}_{Z_k \in \mathcal{D}_k} \mathsf{E}\Big(\sum_{k=n+1}^{N} Z_{n+1} \dots Z_k \langle h, X_k \rangle \,\Big|\, \mathcal{F}_n\Big) = u_n(\langle h, X \rangle) = \varphi_h(G_n(X)), \quad h \in \mathbb{R}^d.$$

Hence, $\mathsf{P}(\varphi_h(G'_n) = \varphi_h(G_n(X)) \ \forall h \in \mathbb{Q}^d) = 1$, which yields the result. □

Example 3.4 (Risk Metrics). Let $\mathcal{D}$ be the dynamic Weighted V@R corresponding to a positive measure μ with $\mu((0,1]) \le 1$ such that the static Weighted V@R ρ_μ is finite on Gaussian random variables. Let X be a d-dimensional process with

$$\mathsf{Law}(X_n \,|\, \mathcal{F}_{n-1}) = \mathcal{N}(0, C_{n-1}), \quad n = 1, \dots, N,$$

where the covariance process C is adapted and satisfies the recurrent relation

$$C_n = \alpha C_{n-1} + (1-\alpha) R_n, \quad R_n^{ij} = X_n^i X_n^j, \quad n = 1, \dots, N.$$

Here α is a fixed parameter from $(0,1)$.[2]

Define the values γ_n by $\gamma_0 = 0$, $\gamma_{n+1} = \rho_\mu(\xi - \gamma_n(\alpha + (1-\alpha)\xi^2)^{1/2})$, where ξ is a standard normal random variable. Let B_n be the image of the unit ball in $\mathbb{R}^d$ under the map $x \mapsto C_n^{1/2} x$. Then

$$G_n(X) = \gamma_{N-n} B_n, \quad n = 0, \dots, N.$$

In order to prove this relation, let us first check the equality

$$u_n(\langle h, X \rangle) = -\gamma_{N-n} \langle h, C_n h \rangle^{1/2}, \quad h \in \mathbb{R}^d,\ n = 0, \dots, N$$

going backwards from N to 0. Suppose the statement is true for n and let us prove it for $n-1$. We have

$$\begin{aligned}
u_{n-1}(\langle h, X \rangle) &= \operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(\langle h, X_n \rangle - \gamma_{N-n} \langle h, C_n h \rangle^{1/2}) \,|\, \mathcal{F}_{n-1}) \\
&= \langle h, C_{n-1} h \rangle^{1/2} \operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}\Big(Z\Big(\frac{\langle h, X_n \rangle}{\langle h, C_{n-1} h \rangle^{1/2}} - \gamma_{N-n}\Big(\alpha + (1-\alpha)\frac{\langle h, X_n \rangle^2}{\langle h, C_{n-1} h \rangle}\Big)^{1/2}\Big) \Big| \mathcal{F}_{n-1}\Big) \\
&= -\gamma_{N-n} \langle h, C_{n-1} h \rangle^{1/2} u_\mu(\xi + (\alpha + (1-\alpha)\xi^2)^{1/2}) \\
&= -\gamma_{N-n+1} \langle h, C_{n-1} h \rangle^{1/2}.
\end{aligned}$$

[2] This might be called the linear Risk Metrics model. In the true Risk Metrics model, the logarithmic rather than the actual prices follow this relation. However, for short time horizons this difference is not essential.

Here ξ is a standard normal random variable; in the third equality we used Lemma 2.5; on the set $\{\langle h, C_{n-1}h\rangle = 0\}$ we pass on directly from the first line to the last one. Thus, $G(X)$ exists and, according to Theorem 3.2,

$$\varphi_h(G_n(X)) = u_n(\langle h, X\rangle) = -\gamma_{N-n}\langle h, C_n h\rangle^{1/2}, \quad h \in \mathbb{R}^d,\ n = 0, \dots, N.$$

Furthermore,

$$\varphi_h(\gamma_{N-n}B_n) = \gamma_{N-n} \min_{z \in B_n^0} \langle h, C_n^{1/2} z\rangle = -\gamma_{N-n}\|C_n^{1/2}h\|, \quad h \in \mathbb{R}^d,\ n = 0, \dots, N,$$

where B_n^0 is the unit ball in $\mathbb{R}^d$. Hence,

$$\mathsf{P}(\varphi_h(G_n(X)) = \varphi_h(\gamma_{N-n}B_n)\ \forall h \in \mathbb{Q}^d) = 1,$$

which yields the desired equality. □

`Remarks`. (i) Clearly, the sequence (γ_n) is strictly increasing and tends to infinity. Furthermore,

$$\lim_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} = \rho_\lambda(\sqrt{\alpha + (1-\alpha)\xi^2}) > 1.$$

This means the the risk at time 0 grows exponentially in N with the exponent $\rho_\lambda(\sqrt{\alpha + (1-\alpha)\xi^2})$. For the values $\alpha = 0.94$, $\lambda = 0.05$, $\lambda = 0.025$, and $\lambda = 0.01$, the value of $\rho_\lambda(\sqrt{\alpha + (1-\alpha)\xi^2})$ equals 1.13, 1.16, and 1.20, respectively.

(ii) If $\mathcal{D}$ is the same as in the above example and (X_n) are independent identically distributed random variables, then $G_n(X) = (N-n)G$, where $G = \mathrm{cl}\{\mathsf{E}_\mathsf{Q} X_n : \mathsf{Q} \in \mathcal{D}_\mu\}$ and $\mathcal{D}_\mu$ is given by (2.4). In particular, the risk at time 0 grows linearly in N. □

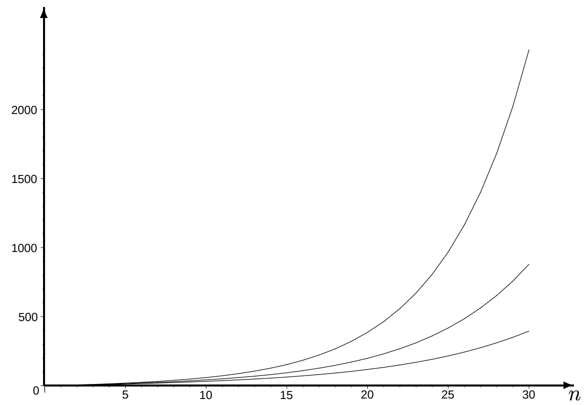

Figure 1. The values γ_n for $\alpha = 0.94$ and $\lambda = 0.05$ (lower plot), $\lambda = 0.025$ (middle plot), $\lambda = 0.01$ (upper plot)

4 Extreme Systems

Definition 4.1. Let X be a one-dimensional $(\mathcal{F}_n)$-adapted process. The *extreme system* corresponding to X is $\mathcal{X}(X) = (\mathcal{X}_n(X))_{n=1,\dots,N}$,

$$\mathcal{X}_n(X) = \operatorname*{argessmin}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(X_n + u_n(X)) \mid \mathcal{F}_{n-1}).$$

If $X_n \in L^1(\overline{\mathcal{D}})$ for any n, then each $\mathcal{X}_n(X)$ is non-empty due to Lemma 2.3 combined with the property $u_n(X) \in L^1(\overline{\mathcal{D}})$, which was checked in the proof of Proposition 2.2. Obviously, $\mathcal{X}_n(X)$ is $\mathcal{F}_{n-1}$-convex and uniformly integrable. Furthermore, for any $\xi \in L^1(\overline{\mathcal{D}})$, the convergence $\mathcal{D}_n \ni Z_k \xrightarrow[k\to\infty]{L^1} Z$ implies the convergence $Z_k\xi \xrightarrow[k\to\infty]{L^1} Z\xi$. Consequently, $\mathcal{X}_n(X)$ is L^1-closed. Thus, $\mathcal{X}(X)$ satisfies all the conditions of a determining system.

Below $G(X;\mathcal{D}')$ denotes the generator of a process X corresponding to a determining system $\mathcal{D}'$.

Theorem 4.2. *If $X_n \in L^1(\overline{\mathcal{D}})$ for any n, then*

$$G_n(X;\mathcal{X}(\langle h, X\rangle)) = \operatorname*{argmin}_{x\in G_n(X)}\langle h, x\rangle, \quad n = 0,\dots,N,\ h\in\mathbb{R}^d.$$

Proof. Fix $h \in \mathbb{R}^d$. Denote the left-hand side of the above equality by H_n and its right-hand side by H_n'. Let us first prove the inclusion $H_n \subseteq H_n'$ going backwards from N to 0. Suppose the inclusion is true for n and let us prove it for $n-1$. Denote

$$L_n = \{x\in\mathbb{R}^d : \langle h, x\rangle = \varphi_h(G_n(X))\} = \{x\in\mathbb{R}^d : \langle h, x\rangle = u_n(\langle h, X\rangle)\},$$

so that $H_n' = L_n \cap G_n(X)$. For any $Z \in \mathcal{D}_n$ and any $Y \in L^0(\mathcal{F}_n, H_n)$, we have $Y \in H_n'$ a.s., so that

$$\langle h, \mathsf{E}(Z(X_n+Y)\,|\,\mathcal{F}_{n-1})\rangle = \mathsf{E}(Z(\langle h, X_n\rangle + u_n(\langle h, X\rangle))\,|\,\mathcal{F}_{n-1}) = u_{n-1}(\langle h, X\rangle).$$

Thus, $H_{n-1}\subseteq L_{n-1}$, and consequently, $H_{n-1}\subseteq H_{n-1}'$.

Let us prove that $H_n'\subseteq H_n$ going backwards from N to 0. Suppose this is true for n and let us prove it for $n-1$. Take $\xi\in L^0(\mathcal{F}_{n-1}, H_{n-1}')$. According to Lemma 4.3, there exist $Z\in\mathcal{D}_n$ and $Y\in L^0(\mathcal{F}_n, G_n(X))$ such that $\xi = \mathsf{E}(Z(X_n+Y)\,|\,\mathcal{F}_{n-1})$. It follows from the line

$$\begin{aligned} u_{n-1}(\langle h, X\rangle) &= \langle h, \xi\rangle = \mathsf{E}(Z(\langle h, X_n\rangle + \langle h, Y\rangle)\,|\,\mathcal{F}_{n-1}) \\ &\ge \operatorname*{essinf}_{Z\in\mathcal{D}_n}\mathsf{E}(Z(\langle h, X_n\rangle + u_n(\langle h, X\rangle))\,|\,\mathcal{F}_{n-1}) = u_{n-1}(\langle h, X\rangle)\end{aligned}$$

that $Z\in\mathcal{X}_n(\langle h, X\rangle)$. Moreover, we see that $\langle h, Y\rangle = u_n(\langle h, X\rangle)$ a.e. on $\{Z>0\}$, i.e. $Y\in H_n'$ a.e. on $\{Z>0\}$. Choose an arbitrary $Y'\in L^0(\mathcal{F}_n, H_n')$ (it exists due to Lemma A.2) and set $\widetilde{Y} = YI(Z>0) + Y'I(Z=0)$. Then $\widetilde{Y}\in L^0(\mathcal{F}_n, H_n')\subseteq L^0(\mathcal{F}_n, H_n)$ and $\xi = \mathsf{E}(Z(X_n+\widetilde{Y})\,|\,\mathcal{F}_{n-1})$. Thus, $\xi\in L^0(\mathcal{F}_{n-1}, H_{n-1})$.

According to [22; Th. 5.6], we can choose a family $\xi_k\in L^0(\mathcal{F}_{n-1}, H_{n-1}')$, $k\in\mathbb{N}$ such that the set $(\xi_k(\omega) : k\in\mathbb{N})$ is dense in $H_{n-1}'(\omega)$ for a.e. ω. The above statement shows that $\xi_k(\omega)\in H_{n-1}(\omega)$ for a.e. ω. Thus, we arrive at the inclusion $H_{n-1}'\subseteq H_{n-1}$. □

Lemma 4.3. *Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space, $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, $\mathcal{E}$ be a non-empty $\mathcal{G}$-convex L^1-closed uniformly integrable set, and C be an $\mathcal{F}$-measurable $\mathcal{C}$-valued map such that $\sup_{x\in C}\|x\|\in L^1(\mathcal{E})$. Consider*

$$D = \operatorname*{essconv}_{Z\in\mathcal{E},\,Y\in L^0(\mathcal{F},C)}\mathsf{E}(ZY\,|\,\mathcal{G})$$

and let $X\in L^0(\mathcal{G}, D)$. Then there exist $Z\in\mathcal{E}$ and $Y\in L^0(\mathcal{F}, C)$ such that $X = \mathsf{E}(ZY\,|\,\mathcal{G})$.

Proof. Consider the set $A=\{\mathsf{E}(ZY\,|\,\mathcal{G}): Z\in\mathcal{E},\, Y\in L^0(\mathcal{F},C)\}$. For any $Z_1,Z_2\in\mathcal{E}$, $Y_1,Y_2\in L^0(\mathcal{F},C)$, and $[0,1]$-valued $\mathcal{G}$-measurable λ, we have

$$\begin{aligned}
&\lambda\mathsf{E}(Z_1Y_1\,|\,\mathcal{G})+(1-\lambda)\mathsf{E}(Z_2Y_2\,|\,\mathcal{G})=\mathsf{E}(\lambda Z_1Y_1+(1-\lambda)Z_2Y_2\,|\,\mathcal{G})\\
&=\mathsf{E}\biggl((\lambda Z_1+(1-\lambda)Z_2)\Bigl(\frac{\lambda Z_1}{\lambda Z_1+(1-\lambda)Z_2}\,Y_1+\frac{(1-\lambda)Z_2}{\lambda Z_1+(1-\lambda)Z_2}\,Y_2\Bigr)\Bigm|\mathcal{G}\biggr)\in A,
\end{aligned}$$

so that A is $\mathcal{G}$-convex.

Fix $\eta\in L^\infty(\mathcal{G},\mathbb{R}^d)$ and let us prove that there exists $\xi_*\in A$ such that $\mathsf{E}\langle\eta,\xi_*\rangle=\inf_{\xi\in A}\mathsf{E}\langle\eta,\xi\rangle$. For this, set $\zeta=\inf_{x\in C}\langle\eta,x\rangle$. Approximating η by simple random variables, we see that ζ is $\mathcal{F}$-measurable. By Lemma A.1, the $\mathcal{C}$-valued map $C'=\{x\in C:\langle\eta,x\rangle=\varphi_\eta(C)\}$ is $\mathcal{F}$-measurable. By Lemma A.2, there exists $Y_*\in L^0(\mathcal{G},C')$. By Lemma 2.3 applied to $\mathcal{G}=\text{triv}$, there exists $Z_*\in\operatorname{argmin}_{Z\in\mathcal{E}}\mathsf{E}Z\langle\eta,Y_*\rangle$. Then, for any $Z\in\mathcal{E}$ and $Y\in L^0(\mathcal{F},C)$, we have

$$\mathsf{E}\langle\eta,\mathsf{E}(ZY\,|\,\mathcal{G})\rangle=\mathsf{E}Z\langle\eta,Y\rangle\ge\mathsf{E}Z\langle\eta,Y_*\rangle\ge\mathsf{E}Z_*\langle\eta,Y_*\rangle=\mathsf{E}\langle\eta,\mathsf{E}(Z_*Y_*\,|\,\mathcal{G})\rangle.$$

By the James theorem (see [16]), A is weakly compact. Consequently, A is weakly closed and, hence, L^1-closed.

Suppose that $X\notin A$. The inclusion $\sup_{x\in C}\|x\|\in L^1(\mathcal{E})$ implies that $\sup_{x\in D}\|x\|\in L^1$, so that $X\in L^1$. By the Hahn-Banach theorem, we can find $\eta\in L^\infty(\mathcal{G},\mathbb{R}^d)$ such that $\mathsf{E}\langle\eta,X\rangle<\inf_{\xi\in A}\mathsf{E}\langle\eta,\xi\rangle$. Moreover, we can find a simple η (i.e. taking a finite number of values) with this property. Then, for $\zeta=\inf_{x\in D}\langle\eta,x\rangle$, we have, due to Lemma A.4,

$$\zeta=\operatorname*{essinf}_{Z\in\mathcal{E},\,Y\in L^0(\mathcal{F},C)}\langle\eta,\mathsf{E}(ZY\,|\,\mathcal{G})\rangle=\operatorname*{essinf}_{\xi\in A}\langle\eta,\xi\rangle.$$

Arguing in the same way as in the proof of Lemma 2.3, we show that the essinf here is attained at some $\xi_*\in A$ (here we need the G-convexity of A). Then $\langle\eta,X\rangle\ge\zeta=\langle\eta,\xi_*\rangle$, which contradicts the choice of η. As a result, $X\in A$. □

Example 4.4 (Risk Metrics). Consider the setting of Example 3.4 and assume that C_0 is non-degenerate (then all C_n automatically satisfy this condition). Then

$$\mathcal{X}_n(\langle h,X\rangle)=\{\psi_\mu(F_n(\eta_n))\},\quad h\in\mathbb{R}^d\setminus\{0\},\,n=1,\dots,N \tag{4.1}$$

$$G_n(X;\mathcal{X}(\langle h,X\rangle)=\bigl\{-\gamma_{N-n}\langle h,C_nh\rangle^{-1/2}C_nh\bigr\},\quad h\in\mathbb{R}^d\setminus\{0\},\,n=0,\dots,N. \tag{4.2}$$

Here

$$\psi_\mu(x)=\int_{[x,1]}\lambda^{-1}\mu(d\lambda),\quad x\in(0,1], \tag{4.3}$$

$$\eta_n=\langle h,C_{n-1}h\rangle^{-1/2}(\langle h,X_n\rangle-\gamma_{N-n}(\alpha\langle h,C_{n-1}h\rangle+(1-\alpha)\langle h,X_n\rangle^2)^{1/2}),$$

and F_n is the distribution function of $\xi-\gamma_{N-n}(\alpha+(1-\alpha)\xi^2)^{1/2}$, where ξ is a standard normal random variable and γ_n is defined in Example 3.4.

Let us first prove (4.1). Fix h and n. Due to Example 3.4,

$$\begin{aligned}
&\operatorname*{argessmin}_{Z\in\mathcal{D}_n}\mathsf{E}(Z(\langle h,X_n\rangle+u_n(\langle h,X\rangle))\,|\,\mathcal{F}_{n-1})\\
&=\operatorname*{argessmin}_{Z\in\mathcal{D}_n}\mathsf{E}(Z\langle h,C_{n-1}h\rangle^{1/2}\eta_n\,|\,\mathcal{F}_{n-1})\\
&=\operatorname*{argessmin}_{Z\in\mathcal{D}_n}\mathsf{E}(Z\eta_n\,|\,\mathcal{F}_{n-1}).
\end{aligned}$$

Note that

$$\mathsf{Law}(\eta_n\,|\,\mathcal{F}_{n-1})=\mathsf{Law}(\xi-\gamma_{N-n}(\alpha+(1-\alpha)\xi^2)^{1/2}).$$

Now, the desired statement follows from Lemma 4.5.

Let us prove (4.2). Fix h and n. By Example 4.4, $G_n(X)=-\gamma_{N-n}B_n$, where $B_n=\{x\in\mathbb{R}^d:\langle x,C_n^{-1}x\rangle\le 1\}$. Due to the non-degeneracy of C_n, $\operatorname{argmin}_{x\in B_n}\langle h,x\rangle$ consists of a unique point x_n. This point provides the minimum of the functional $\langle h,x\rangle+\alpha\langle x,C_n^{-1}x\rangle$ with some positive Lagrange multiplier α. Differentiating, we find that $x_n=-\beta C_nh$ with some positive β. The latter value can be found from the condition $\langle x_n,C_n^{-1}x_n\rangle=1$, so that finally $x_n=-\langle h,C_nh\rangle^{-1/2}C_nh$. An application of Theorem 4.2 completes the proof. □

Lemma 4.5. *Let $\Omega,\mathcal{F},\mathsf{P},\mathcal{G},\mu,\mathcal{D}$ be the same as in Lemma 2.5 and $X\in L^1(\mathcal{D})$ be such that $\mathsf{Law}(X\,|\,\mathcal{G})$ is continuous a.s. Then*

$$\operatorname*{argessmin}_{Z\in\mathcal{D}}\mathsf{E}(ZX\,|\,\mathcal{G})=\{\psi_\mu(F(X,\cdot))\},$$

where ψ_μ is given by (4.3) *and $F(x,\omega)=\mathsf{P}(X\le x\,|\,\mathcal{G})(\omega)$.*

`Proof`. Denote $Z_*=\psi_\mu(F(X,\cdot))$ and let us first check the inclusion $Z_*\in\mathcal{D}$. It is easy to see that $\mathsf{Law}(F(X,\cdot)|\mathcal{G})$ is uniform a.s. Thus, it is sufficient to check that, for a $[0,1]$-uniformly distributed random variable ξ, $\mathsf{E}\psi_\mu(\xi)=\mu((0,1])$ and $\mathsf{E}(\psi_\mu(\xi)-x)^+\le\Phi_\mu(x)$ for any $x\in\mathbb{R}_+$. Note that $\psi_\mu(\xi)$ can be written as $\int_{(0,1]}\lambda^{-1}I(\xi\le\lambda)\mu(d\lambda)$. Now, the desired properties follow from the comparison of [10; Th. 4.4] and [10; Th. 4.6].

Denote $\mathsf{Q}_\omega=\mathsf{Law}(X\,|\,\mathcal{G})(\omega)$. Then, for a.e. ω,

$$\mathsf{E}(Z_*X\,|\,\mathcal{G})(\omega)=\int_{\mathbb{R}}x\psi_\mu(\mathsf{Q}_\omega((-\infty,x]))\mathsf{Q}_\omega(dx)=\widetilde{u}_\mu(\mathsf{Q}_\omega)=\operatorname*{essinf}_{Z\in\mathcal{D}}\mathsf{E}(ZX\,|\,\mathcal{G})(\omega),$$

where $\widetilde{u}_\mu$ is the same as in Lemma 2.5. The second equality here follows from [10; Prop. 6.2] and the third one follows from Lemma 2.5.

Finally, take an arbitrary element $Z\in\operatorname{argessmin}_{Z\in\mathcal{D}}\mathsf{E}(ZX\,|\,\mathcal{G})$. Denote $\mathsf{Q}_\omega=\mathsf{Law}(X,Z_*,Z\,|\,\mathcal{G})(\omega)$ and let θ^i denote the projection of $\mathbb{R}^3$ on the i-th axis. For a.e. ω, we have

$$\begin{aligned}
&\mathsf{E}_{\mathsf{Q}_\omega}\theta^1\theta^i=\widetilde{u}_\mu(\mathsf{Law}_{\mathsf{Q}_\omega}\theta^1)=u_\mu(\theta^1),\quad i=2,3,\\
&\mathsf{E}_{\mathsf{Q}_\omega}\theta^i=\mu((0,1]),\quad i=2,3,\\
&\mathsf{E}_{\mathsf{Q}_\omega}(\theta^i-x)^+\le\Phi_\mu(x),\quad x\in\mathbb{Q}_+,\ i=2,3.
\end{aligned}$$

The first equality here follows from Lemma 2.5. As the functions $\mathsf{E}_{\mathsf{Q}_\omega}(\theta^2-x)^+$ and $\Phi_\mu(x)$ are continuous in x, the above inequality holds for any $x\in\mathbb{R}_+$. As $\mathsf{Law}_{\mathsf{Q}_\omega}\theta^1$ is continuous for a.e. ω, it follows from [10; Prop. 2.7] that, for a.e. ω, $\theta^2=\theta^3$ Q_ω-a.s. This means that $Z=Z_*$ a.s. □

5 Capital Allocation

The definition below is a straightforward dynamic extension of the static definition given by Delbaen [12; Sect. 9].

Definition 5.1. Let $X = (X^1, \dots, X^d)$ be a d-dimensional $(\mathcal{F}_n)$-adapted process. A *utility allocation* between $X^1, \dots, X^d$ is a d-dimensional adapted process Y such that

$$\sum_{i=1}^{d} Y_n^i = u_n\Big(\sum_{i=1}^{d} X^i\Big), \quad n = 0, \dots, N,$$

$$\sum_{i=1}^{d} h^i Y_n^i \ge u_n\Big(\sum_{i=1}^{d} h^i X^i\Big), \quad h \in \mathbb{R}_+^d,\ n = 0, \dots, N.$$

A *capital allocation* is a utility allocation with the minus sign.

A typical financial interpretation is as follows: X^i is the stream of cash flows received by the i-th component of a firm and $-Y^i$ is the part of the risk of the whole firm carried by the i-th component.

Theorem 5.2. **(i)** (*Geometric representation*) *If the generator of X exists, then an adapted process Y is a utility allocation if and only if $Y_n \in \operatorname{argmin}_{x \in G_n(X)} \langle e, x\rangle$ a.s. for any n, where $e = (1, \dots, 1)$.*

(ii) (*Probabilistic representation*) *If $\|X_n\| \in L^1(\overline{\mathcal{D}})$ for any n, then an adapted process Y is a utility allocation if and only if $Y_n \in G_n(X; \mathcal{X}_n(\langle e, X\rangle))$ a.s. for any n.*

`Proof`. **(i)** This statement follows from the line

$$\Big\{x \in \mathbb{R}^d : \sum_{i=1}^{d} x^i = u_n\Big(\sum_{i=1}^{d} X^i\Big) \text{ and } \sum_{i=1}^{d} h^i x^i \ge u_n\Big(\sum_{i=1}^{d} h^i X^i\Big)\ \forall h \in \mathbb{R}_+^d\Big\}$$

$$= \{x \in \mathbb{R}^d : \langle e, x\rangle = \varphi_e(G_n(X)) \text{ and } \langle h, x\rangle \ge \varphi_h(G_n(X))\ \forall h \in \mathbb{R}_+^d\}$$

$$= \operatorname*{argmin}_{x \in G_n(X)} \langle e, x\rangle, \quad n = 0, \dots, N,$$

where the first equality is a consequence of Theorem 3.2.

(ii) This statement follows from (i) and Theorem 4.2. □

Example 5.3 (Risk Metrics). Consider the setting of Example 3.4 and assume that C_0 is non-degenerate. It follows from Example 4.4 that the utility allocation is unique and has the form $Y_n = -\gamma_{N-n}\langle e, C_n e\rangle^{-1/2} C_n e$. □

`Remark`. In typical situations (as in the above example), each $\mathcal{X}_n(\sum_i X^i)$ consists of a unique element Z_n^*. As follows from Corollary 3.3, in this case the utility allocation is unique and is given by

$$Y_n = \mathsf{E}\Big(\sum_{k=n+1}^{N} Z_{n+1}^* \dots Z_k^* X_k \,\Big|\, \mathcal{F}_n\Big), \quad n = 0, \dots, N$$

(here we assume that $\|X_n\| \in L^1(\overline{\mathcal{D}})$ for any n). If moreover $\mathcal{D}$ is probabilistic, then

$$Y_n = \mathsf{E}_{\mathsf{Q}_*}\Big(\sum_{k=n+1}^{N} X_k \,\Big|\, \mathcal{F}_n\Big), \quad n = 0, \dots, N,$$

where $\mathsf{Q}_* = Z_1^* \dots Z_N^* \mathsf{P}$. The measure Q_* might be called the *extreme measure* corresponding to $\sum_i X^i$. □

6 Risk Contribution

Definition 6.1. Let X, Y be one-dimensional $(\mathcal{F}_n)$-adapted processes. The *utility contribution* of X to Y is the process

$$u_n^c(X;Y) = \operatorname{essinf}\{W_n^1 : W \text{ is a utility allocation between } X \text{ and } Y-X\}, \quad n = 0, \dots, N.$$

The *risk contribution* is the utility contribution with the minus sign.

By $u(X;\mathcal{D}')$ we denote the coherent utility of a process X corresponding to a determining system $\mathcal{D}'$.

Theorem 6.2. **(i)** (*Geometric representation*) *If the generator of* (X, Y) *exists, then*

$$u_n^c(X,Y) = \min\{x : (x, \min\{y' : (x', y') \in G_n(X,Y)\}) \in G_n(X,Y)\}, \quad n = 0, \dots, N.$$

(ii) (*Probabilistic representation*) *If* $X_n, Y_n \in L^1(\overline{\mathcal{D}})$ *for any* n, *then*

$$u_n^c(X;Y) = u_n(X;\mathcal{X}(Y)), \quad n = 0, \dots, N.$$

(iii) (*Analytic representation*) *If the generator of* (X, Y) *exists, then*

$$u_n^c(X;Y) = (\text{a.s.}) \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(u_n(Y + \varepsilon X) - u_n(Y)), \quad n = 0, \dots, N.$$

Proof. **(i)** Denote

$$E_n = \operatorname*{argmin}_{(x,y) \in G_n(X,Y)} y, \quad n = 0, \dots, N.$$

It follows from Theorem 5.2 (i) that an adapted process W is a utility allocation between X and $Y-X$ if and only if $W_n \in E_n$ a.s. for any n. By Lemma A.1, E_n is $\mathcal{F}_n$-measurable. Taking

$$W_n = \operatorname*{argmin}_{(x,y) \in E_n} x, \quad n = 0, \dots, N$$

yields the desired statement.

(ii) By Theorem 4.2, $G_n(X, Y; \mathcal{X}(Y)) = E_n$. By Theorem 3.2, $u_n(X;\mathcal{X}(Y)) = \min\{x : (x, y) \in E_n\}$. It remains to use (i).

(iii) By Theorem 3.2, $u_n(\varepsilon X + Y) = \varphi_{(\varepsilon,1)}(G_n(X,Y))$. Now, the desired statement follows from Lemma 6.3. □

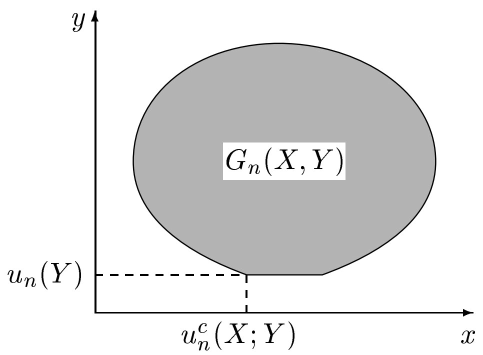

Figure 2. Geometric representation of the utility contribution

Lemma 6.3. *For a convex compact C in $\mathbb{R}^2$, we have*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}(\varphi_{(\varepsilon,1)}(C)-\varphi_{(0,1)}(C))=\min\{x:(x,\min\{y':(x',y')\in C\})\in C\}.$$

`Proof`. Denote $b=\min\{y:(x,y)\in C\}$, $a=\min\{x:(x,b)\in C\}$. Take

$$(a(\varepsilon),b(\varepsilon))\in\operatorname*{argmin}_{(x,y)\in C}\langle(\varepsilon,1),(x,y)\rangle.$$

It is clear that $a(\varepsilon)\le a$, $b(\varepsilon)\ge b$, and $a(\varepsilon)\xrightarrow[\varepsilon\downarrow 0]{} a$, $b(\varepsilon)\xrightarrow[\varepsilon\downarrow 0]{} b$. Furthermore,

$$\varepsilon a(\varepsilon)+b(\varepsilon)\le\varepsilon a+b,$$

which implies that

$$0\le\varepsilon^{-1}(b(\varepsilon)-b)\le a-a(\varepsilon).$$

As a result,

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}(\varphi_{(\varepsilon,1)}(C)-\varphi_{(0,1)}(C))=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}(\varepsilon a(\varepsilon)+b(\varepsilon)-b)=a.$$

Example 6.4 (Risk Metrics). Consider the setting of Example 3.4 with the d-dimensional process X replaced by the 2-dimensional process (X,Y). We assume that C_0 is non-degenerate. According to Example 5.3,

$$u_n^c(X;Y)=-\gamma_{N-n}\langle e_2,C_ne_2\rangle^{-1/2}\langle e_1,C_ne_2\rangle=-\gamma_{N-n}\frac{C_n^{12}}{(C_n^{22})^{1/2}}=-\gamma_{N-n}\frac{\mathsf{cov}(X_n,Y_n\,|\,\mathcal{F}_{n-1})}{(\mathsf{var}(Y_n\,|\,\mathcal{F}_{n-1}))^{1/2}},$$

where $e_1=(1,0)$, $e_2=(0,1)$, and var denotes the variance. □

`Remark`. In typical situations (as in the above example), each $\mathcal{X}_n(Y)$ consists of a unique element Z_n^*. As follows from Proposition 2.2, in this case

$$u_n^c(X;Y)=\mathsf{E}\biggl(\sum_{k=n+1}^N Z_{n+1}^*\dots Z_k^*X_k\biggm|\mathcal{F}_n\biggr),\quad n=0,\dots,N$$

(here we assume that $X_n,Y_n\in L^1(\overline{\mathcal{D}})$ for any n). If moreover $\mathcal{D}$ is probabilistic, then

$$u_n^c(X;Y)=\mathsf{E}_{\mathsf{Q}_*}\biggl(\sum_{k=n+1}^N X_k\biggm|\mathcal{F}_n\biggr),\quad n=0,\dots,N,$$

where $\mathsf{Q}_*=Z_1^*\dots Z_N^*\mathsf{P}$. □

Appendix

Denote by $\mathcal{C}$ the set of non-empty convex compacts in $\mathbb{R}^d$. It is well known that $\mathcal{C}$ endowed with the Hausdorff metrics

$$\rho(C_1,C_2)=\sup_{x_n\in C_n}\|x_1-x_2\|$$

is a Polish space.

Lemma A.1. *The map*

$$A : \mathbb{R}^d \times \mathcal{C} \ni (h, C) \longmapsto \operatorname*{argmin}_{x\in C}\langle h, x\rangle \in \mathcal{C}$$

is measurable.

Proof. For any n, the map

$$A_n : \mathbb{R}^d \times \mathcal{C} \ni (h, C) \longmapsto \{x \in C : \langle h, x\rangle \le \varphi_h(C) + 1/n\} \in \mathcal{C}$$

is continuous and, therefore, measurable. Furthermore, $A(h, C) = \bigcap_{n=1}^{\infty} A_n(h, C)$, so that $\varphi_g \circ A = \lim_n \varphi_g \circ A_n$ is measurable for any $g \in \mathbb{R}^d$. As the functions φ_g, $g \in \mathbb{R}^d$ are continuous and separate points, $\mathcal{B}(\mathcal{C}) = \sigma(\varphi_g : g \in \mathbb{R}^d)$ (see [29; Ch. 1, § 1]). Hence, A is measurable. □

Lemma A.2. *Let C be a $\mathcal{C}$-valued random element and X be a d-dimensional random vector. Then the set $\{X \in C\}$ is measurable. Furthermore, there exists a d-dimensional random vector Y such that $Y \in C$ a.s.*

Proof. The first statement follows from the equality

$$\{X \in C\} = \bigcap_{h\in\mathbb{Q}^d}\{\langle h, X\rangle \ge \varphi_h(C)\}.$$

In order to prove the second one, consider the set $A = \{(\omega, x) : x \in C(\omega)\}$. In view of the representation

$$A = \bigcap_{h\in\mathbb{Q}^d}\{(\omega, x) : \langle h, x\rangle \ge \varphi_h(C(\omega))\},$$

A is measurable. An application of the measurable selection theorem completes the proof. □

Definition A.3. Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of $\mathbb{R}^d$-valued random vectors. The *essential closed convex hull* of $(X_\lambda)_{\lambda\in\Lambda}$ is a $\mathcal{C}$-valued random element C with the properties:

(a) for any λ, $X_\lambda \in C$ a.s.

(b) if C' is another random element with this property, then $C \subseteq C'$ a.s.

We use the notation $C = \operatorname{essconv}_{\lambda\in\Lambda} X_\lambda$.

Lemma A.4. *If* $\operatorname{esssup}_\lambda \|X_\lambda\| < \infty$, *then* $\operatorname{essconv}_\lambda X_\lambda$ *exists. Moreover,*

$$\varphi_h\bigl(\operatorname*{essconv}_{\lambda\in\Lambda} X_\lambda\bigr) = \operatorname*{essinf}_{\lambda\in\Lambda}\langle h, X_\lambda\rangle, \quad h \in \mathbb{R}^d. \tag{a.2}$$

Proof. The random set

$$C = \bigcap_{h\in\mathbb{Q}^d}\bigl\{x \in \mathbb{R}^d : \langle h, x\rangle \ge \operatorname*{essinf}_{\lambda\in\Lambda}\langle h, X_\lambda\rangle\bigr\}$$

is measurable and satisfies condition (a). If C' is another random set with these properties, then, by the finite-dimensional Hahn-Banach theorem,

$$C' = \bigcap_{h\in\mathbb{Q}^d}\bigl\{x \in \mathbb{R}^d : \langle h, x\rangle \ge \varphi_h(C')\bigr\}.$$

For any λ, we have $\varphi_h(C') \le \langle h, X_\lambda\rangle$ a.s., so that $\varphi_h(C') \le \operatorname{essinf}_\lambda\langle h, X_\lambda\rangle$. This proves the existence of essconv as well as (a.2) for $h \in \mathbb{Q}^d$. Equality (a.2) for an arbitrary h is obtained by passing on to the limit. □

References

[1] *P. Artzner, F. Delbaen, J.-M. Eber, D. Heath.* Thinking coherently. Risk, **10** (1997), No. 11, p. 68–71.

[2] *P. Artzner, F. Delbaen, J.-M. Eber, D. Heath.* Coherent measures of risk. Mathematical Finance, **9** (1999), No. 3, p. 203–228.

[3] *P. Artzner, F. Delbaen, J.-M. Eber, D. Heath., H. Ku.* Coherent multiperiod risk adjusted values and Bellman's principle. Preprint, available at: `http://www.math.ethz.ch/~delbaen`.

[4] *P. Cheridito, F. Delbaen, M. Kupper.* Coherent and convex monetary risk measures for bounded càdlàg processes. Stochastic Processes and Their Applications, **112** (2004), No. 1, p. 1–22.

[5] *P. Cheridito, F. Delbaen, M. Kupper.* Coherent and convex monetary risk measures for unbounded càdlàg processes. Finance and Stochastics, **10** (2006), No. 3, p. 427–448.

[6] *P. Cheridito, F. Delbaen, M. Kupper.* Dynamic monetary risk measures for bounded discrete-time processes. Electronic Journal of Probability, **11** (2006), p. 57–106.

[7] *P. Cheridito, M. Kupper.* Composition of time-consistent dynamic monetary risk measures in discrete time. Preprint, available at: `http://www.math.ethz.ch/~kupper`.

[8] *A.S. Cherny.* Pricing with coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[9] *A.S. Cherny.* Equilibrium with coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[10] *A.S. Cherny.* Weighted V@R and its properties. Finance and Stochastics, **10** (2006), No. 3, p. 367–393.

[11] *A.S. Cherny, D.B. Madan.* Coherent measurement of factor risks. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[12] *F. Delbaen.* Coherent monetary utility functions. Preprint, available at `http://www.math.ethz.ch/~delbaen` under the name "Pisa lecture notes".

[13] *M. Denault.* Coherent allocation of risk capital. Journal of Risk, **4** (2001), No. 1, p. 1–34.

[14] *K. Detlefsen, G. Scandolo.* Conditional and dynamic convex risk measures. Finance and Stochastics, **9** (2005), No. 4, p. 539–561.

[15] *T. Fischer.* Risk capital allocation by coherent risk measures based on one-sided moments. Insurance: Mathematics and Economics, **32** (2003), No. 1, p. 135–146.

[16] *K. Floret.* Weakly compact sets. Lecture Notes in Mathematics, **801** (1980).

[17] *H. Föllmer, I. Penner.* Convex risk measures and the dynamics of their penalty functions. Preprint, available at: `http://wws.mathematik.hu-berlin.de/~foellmer`.

[18] *H. Föllmer, A. Schied.* Convex measures of risk and trading constraints. Finance and Stochastics, **6** (2002), No. 4, p. 429–447.

[19] *M. Frittelli, E. Rosazza Gianin.* Putting order in risk measures. Journal of Banking and Finance, **26** (2002), No. 7, p. 1473–1486.

[20] *M. Frittelli, E. Rosazza Gianin.* Dynamic convex risk measures. In: G. Szegö (Ed.). Risk measures for the 21st century. Wiley, 2004.

[21] *M. Frittelli, G. Scandolo.* Risk measures and capital requirements for processes. Preprint, available at: `http://www.dmd.unifi.it/scandolo`.

[22] *C.J. Himmelberg.* Measurable relations. Fund. Math., **87** (1975), p. 53–72.

[23] *A. Jobert, L.C.G. Rogers.* Pricing operators and dynamic convex risk measures. Preprint, available at: `http://www.statslab.cam.ac.uk/~chris`.

[24] *M. Kalkbrenner.* An axiomatic approach to capital allocation. Mathematical Finance, **15** (2005), No. 3, p. 425–437.

[25] *L. Overbeck.* Allocation of economic capital in loan portfolios. In: W. Härdle, G. Stahl (Eds.). Measuring risk in complex stochastic systems. Lecture Notes in Statistics, **147** (1999).

[26] *F. Riedel.* Dynamic coherent risk measures. Stochastic Processes and their Applications, **112** (2004), No. 2, p. 185–200.

[27] *B. Roorda, J.M. Schumacher, J. Engwerda.* Coherent acceptability measures in multiperiod models. Mathematical Finance, **15** (2005), No. 4, p. 589–612.

[28] *D. Tasche.* Expected shortfall and beyond. Journal of Banking and Finance, **26** (2002), No. 7, p. 1519–1533.

[29] *N.N. Vakhania, V.I. Tarieladze, S.A. Chobanyan.* Probability distributions on Banach spaces. Dordrecht, 1987.

[30] *S. Weber.* Distribution-invariant risk measures, information, and dynamic consistency. Mathematical Finance, **16** (2006), p. 419–441.