EQUILIBRIUM WITH COHERENT RISK

A.S. Cherny

Moscow State University,
Faculty of Mechanics and Mathematics,
Department of Probability Theory,
119992 Moscow, Russia.
`E-mail: alexander.cherny@gmail.com`
`Webpage: http://mech.math.msu.su/~cherny`

Abstract. This paper is the continuation of [6] and deals with further applications of coherent risk measures to problems of finance.

First, we study the portfolio optimization problem in two forms.

Furthermore, the results obtained are applied to the optimality pricing. Three forms of this technique are considered.

Finally, we study the equilibrium problem both in the unconstrained and in the constrained forms. We establish the equivalence between the global and the competitive optima and give a dual description of the equilibrium. Moreover, we provide an explicit geometric solution of the constrained equilibrium problem.

Most of the results are presented on two levels: for a general model, the results have a probabilistic form; for a static model with a finite number of assets, the results have a geometric form.

Key words and phrases. Coherent risk measure, equilibrium, extreme measure, generator, liquidity, No Better Choice, optimality pricing, portfolio optimization.

1 Introduction

1. Goal of the paper. In this paper, we consider applications of coherent risk measures to

- portfolio optimization;
- optimality pricing;
- equilibrium.

2. Optimization. The optimization problem is considered in two forms. First, we study what we call the *agent-independent optimization*. It is, in fact, the Markowitz-type optimization problem with variance replaced by a coherent risk measure[1], i.e. a problem of the form

$$\begin{cases} \mathsf{E}X \longrightarrow \max, \\ \rho(X) \le c. \end{cases} \tag{1.1}$$

[1]It has been clear from the outset that variance is not a very good measure of risk because high profits are penalized in the same way as high losses. In [21], Markowitz proposed a way to overcome this problem by considering semivariance $\|(X-\mathsf{E}X)^-\|_{L^2}$ instead of variance. The function $\rho(X) = -\mathsf{E}X + \alpha\|(X-\mathsf{E}X)^-\|_{L^2}$ with $0 \le \alpha \le 1$ is, in fact, an example of a coherent risk measure (see [14]). Thus, in essence, semivariance is a particular case of the coherent risk. However, there exist more convenient coherent risk measures (a comparison of different ones is given in [8]).

Here X means the discounted P&L earned by a portfolio and ρ is a coherent risk measure (P&L means the Profit&Loss, i.e. the difference between the terminal wealth and the initial wealth). Let us remark that this problem was considered in [1], [25], [26][2]. As opposed to these papers, we have at our disposal the notion of a *generator* introduced in [6]. In terms of generators, we are able to provide an explicit geometric representation of the optimal portfolio (see Figures 1, 2). The model we are considering takes into account such market imperfections as cone portfolio constraints, transaction costs, and the ambiguity of the historic probability measure.

Problem (1.1) is the optimization problem for an investor who has some sum of money and invests it in some securities. However, it is possible that already before this investment he/she possesses a capital with a random terminal value W (it might have a financial or a non-financial structure; for example, it might be the terminal wealth of a firm producing some goods). If the investor applies a trading strategy providing a P&L X, then he/she passes from W to $X+W$. If the investor is trying to minimize the risk, he/she is faced with the problem

$$\rho(X+W) \longrightarrow \min, \tag{1.2}$$

which we call the *single-agent optimization problem.* Note that this coincides with the superreplication problem for the NGD pricing (see [6; Subsect. 3.6]). We provide a geometric solution of (1.2) for a model with portfolio constraints; see Figure 4 (in [6; Subsect. 3.6] we gave a geometric solution for the case with no constraints). Also, in [7; Sect. 5], we provide sufficient conditions for the uniqueness of a solution of (1.1) and (1.2).

3. Optimality pricing. In [6], we considered two forms of the pricing technique based on coherent risks: utility-based NGD and RAROC-based NGD. Both of them yield interval estimates of a price. However, pricing often means providing a point estimate. In this paper, we combine the RAROC-based NGD with problem (1.1), thus obtaining a pricing technique, which provides a point estimate. We call it *agent-independent No Better Choice pricing* (henceforth, No Better Choice will be abbreviated as NBC). Further research (see [9]) shows that the agent-independent NBC pricing is, in fact, the analog of the empirical asset pricing[3] with the expected utility replaced by the coherent one.

The agent-independent optimality pricing provides a unique price of a contingent claim for any agent. Thus, it does not take into account personal preferences and endowments of different agents. We also introduce another form of the optimality pricing, which takes into account these parameters, thus providing different fair prices of the same contingent claim for different agents. We call it *single-agent optimality pricing.* It is based on optimization problem (1.2) and might be considered as the analog of the classical utility-indifference pricing (also known in the finance literature as reservation pricing) with the classical expected utility replaced by the coherent one.

Then we introduce one more technique called the *multi-agent optimality pricing.* The idea is as follows. We have a contingent claim and several agents, each having his/her own endowment and employing his/her own coherent utility. A price is said to be fair if it provides no trading opportunity, which would allow each agent to increase his/her utility (this is a modified form of the idea proposed in [5]). As an outcome, this technique typically provides a whole interval of prices, which are fair for this group of agents (in

[2]If ρ is defined by a finite number of probabilistic scenarios, i.e. $\rho(X) = -\min_{n=1,\dots,N} \mathsf{E}_{\mathsf{Q}_n} X$, then (1.1) coincides with the optimization problem considered in the generalized Neyman–Pearson lemma (see [20; Ch. 3]).

[3]Empirical asset pricing based on the classical expected utility is very popular in the modern academic finance literature; see, in particular, [2], [16], [27], [28].

fact, this interval is the convex hull of fair prices for these agents produced by the previous technique).

4. Equilibrium. One of the basic results of the classical economic theory is the equivalence between the global optimum (known also as the "Soviet-type optimum") and the competitive optimum (known also as the "western-type optimum"); see [19]. This result is established within the framework of the expected utility.

In the present paper, we establish an analog of this result within the coherent utility framework. This is done for two types of equilibrium: for the unconstrained one (Theorem 4.7) and for the constrained one (Theorem 4.13). A very important feature of coherent utility (which is not shared by the expected utility) is that it admits a rich duality theory. Thus, besides establishing the equivalence between different types of equilibrium, we also provide its dual description.

Moreover, for the constrained equilibrium problem, we are able to provide an explicit geometric solution based on generators (see Figure 10). It yields the equilibrium price as well as the equilibrium portfolios of the agents.

Some of our results on equilibrium are, in fact, extensions of results of the paper [15] by Heath and Ku. Let us also mention the papers [3] and [18], which study the two-agent unconstrained equilibrium and, in particular, give the explicit solution of this problem for some particular cases. The results of Subsection 4.1 are close to those obtained by Filipovic and Kupper [12], [13]; the major differences are that these papers deal with convex risk measures (that are more general than coherent ones) but we are considering the possibility that each participant can trade in the market.

5. Structure of the paper. Section 2 deals with the optimization problem. In Subsections 2.1 and 2.3, we consider the two settings described above. In Subsection 2.2, the obtained results are applied to the problem of finding the optimal structure of a firm consisting of several units. In Subsection 2.4, we apply the obtained results to the study of the liquidity effects in the framework of the NGD pricing considered in [6].

Section 3 is related to the optimality pricing. Subsections 3.1, 3.2, and 3.3 correspond to the three techniques described above.

Section 4 deals with equilibrium. Subsections 4.1 and 4.2 are, in fact, duals of each other: they consider the unconstrained and the constrained equilibria, respectively.

Altogether, there are seven pricing techniques based on coherent risk proposed in [6] and in the present paper. They are compared in the final Section 5.

2 Optimization

2.1 Agent-Independent Optimization

We consider the model of [6; Subsect. 3.2]. Thus, we are given a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, a convex weakly compact set $\mathcal{RD} \subseteq \mathcal{P}$, an L^1-closed convex set $\mathcal{PD} \subseteq \mathcal{RD}$, and a convex $\mathcal{RD}$-adapted (see [6; Def. 3.2]) set $A \subseteq L^0$. Let us introduce the notation $\mathsf{E}_{\mathcal{PD}} X = \inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E}_{\mathsf{Q}} X$, $u(X) = \inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_{\mathsf{Q}} X$, $\rho(X) = -u(X)$ (we understand $\mathsf{E}_{\mathsf{Q}} X$ according to the convention of [6; Def. 2.3]).

Problem (agent-independent optimization): The problem is

$$\begin{cases} \mathsf{E}_{\mathcal{PD}} X \longrightarrow \max, \\ X \in A, \ \rho(X) \le c, \end{cases}$$

where $c \in \mathbb{R}_+$. Clearly, if A is a cone, then this problem is obviously equivalent to the problem of finding

$$R_* = \sup_{X \in A} \mathrm{RAROC}(X)$$

and

$$X_* = \operatorname*{argmax}_{X \in A} \mathrm{RAROC}(X),$$

where

$$\mathrm{RAROC}(X) = \begin{cases} +\infty & \text{if } \mathsf{E}_{\mathcal{PD}} X > 0 \text{ and } u(X) \geq 0, \\ \dfrac{\mathsf{E}_{\mathcal{PD}} X}{\rho(X)} & \text{otherwise} \end{cases}$$

with the convention $\frac{0}{0} = 0$, $\frac{\infty}{\infty} = 0$.

The only statement we can make at this level of generality is that

$$R_* = \inf\Big\{R > 0 : \Big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Big) \cap \mathcal{R} \neq \emptyset\Big\},$$

where $\mathcal{R}$ is the set of risk-neutral measures (see [6; Def. 3.1]). This follows from [6; Cor. 3.10]. Of course, in general X_* need not exist.

We will now study the problem for a static model with a finite number of assets. Let $\mathcal{PD} \subseteq \mathcal{RD} \subseteq \mathcal{P}$ be convex sets, $A = \{\langle h, S_1 - S_0\rangle : h \in H\}$, where $S_0 \in \mathbb{R}^d$, $S_1^1, \dots, S_1^d \in L_w^1(\mathcal{RD})$, and $H \subseteq \mathbb{R}^d$ is a closed convex cone. From the financial point of view, S_n^i is the discounted price of the i-th asset at time n and H is a convex portfolio constraint. Let us introduce the notation (see Figure 1)

$$\begin{aligned} H^* &= \{x \in \mathbb{R}^d : \forall h \in H,\ \langle h, x\rangle \geq 0\}, \\ E &= \mathrm{cl}\{\mathsf{E}_{\mathsf{Q}} S_1 : \mathsf{Q} \in \mathcal{PD}\}, \\ G &= \mathrm{cl}\{\mathsf{E}_{\mathsf{Q}} S_1 : \mathsf{Q} \in \mathcal{RD}\}, \\ D &= G + H^*, \end{aligned} \tag{2.1}$$

where "cl" denotes the closure, and let D° denote the relative interior of D. (The set G is the generator for S_1 and u.) The sets E and G are convex compacts, while D is convex and closed. Note that, for $h \in H$,

$$\mathsf{E}_{\mathcal{PD}} \langle h, S_1 - S_0\rangle = \inf_{x \in E} \langle h, x - S_0\rangle, \tag{2.2}$$

$$u(\langle h, S_1 - S_0\rangle) = \inf_{x \in G} \langle h, x - S_0\rangle = \inf_{x \in D} \langle h, x - S_0\rangle. \tag{2.3}$$

We will assume that $S_0 \in D^\circ \setminus E$. This assumption is justified economically. Indeed, if $S_0 \in E$, then, in view of (2.2), $\mathrm{RAROC}(X) = 0$ for any $X \in A$; if $S_0 \notin D^\circ$, then, in view of (2.3), there exists $X \in A$ with $\mathrm{RAROC}(X) = \infty$ (provided that E belongs to the relative interior of G).

For $\lambda > 0$, we denote $E(\lambda) = S_0 - \lambda(E - S_0)$ and set $\lambda_* = \sup\{\lambda > 0 : E(\lambda) \cap D \neq \emptyset\}$,

$$N = \{h \in H : \exists a \in \mathbb{R} : \forall x \in E(\lambda_*), \forall y \in D,\ \langle h, x\rangle \leq a \leq \langle h, y\rangle \text{ and } \forall y \in D^\circ,\ \langle h, y\rangle > a\}.$$

Note that $\lambda_* > 0$ due to the condition $S_0 \in D^\circ \setminus E$. Furthermore, N is non-empty provided that $\lambda_* < \infty$. In the case, where $\lambda_* = \infty$, we set $N = H$.

Theorem 2.1. *We have* $R_* = \lambda_*^{-1}$ *and* $\operatorname{argmax}_{h \in H} \mathrm{RAROC}(\langle h, S_1 - S_0\rangle) = N$.

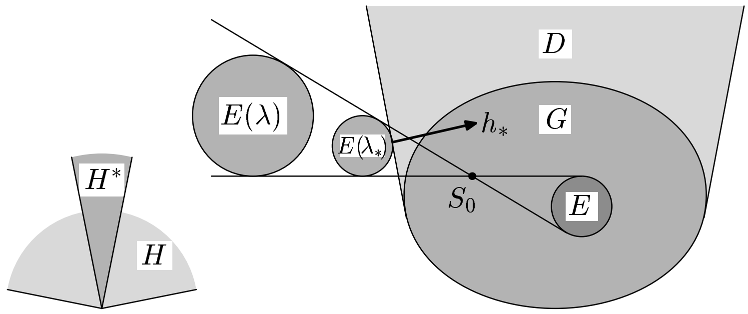

Figure 1. Solution of the optimization problem. Here h_* is an optimal h.

`Proof`. We will prove the statement for the case $\lambda_* < \infty$. The proof for the case $\lambda_* = \infty$ is similar. Take $T \in E(\lambda_*) \cap D$ and set $U = S_0 - \lambda_*^{-1}(T - S_0)$.

If $h \in N$, then

$$\mathrm{RAROC}(\langle h, S_1 - S_0\rangle) = \frac{\inf_{x\in E}\langle h, x - S_0\rangle}{-\inf_{x\in D}\langle h, x - S_0\rangle} = \frac{\langle h, U - S_0\rangle}{-\langle h, T - S_0\rangle} = \lambda_*^{-1}.$$

If $h \in H \setminus N$, then there are three possibilities:

1) h is orthogonal to the smallest affine space containing D;
2) $\sup_{x\in E(\lambda_*)}\langle h, x\rangle > \langle h, T\rangle$;
3) $\inf_{x\in D}\langle h, x\rangle < \langle h, T\rangle$.

In the first case, $\mathrm{RAROC}(\langle h, S_1 - S_0\rangle) = 0$. In the second case,

$$\inf_{x\in E}\langle h, x - S_0\rangle < \langle h, U - S_0\rangle, \qquad \inf_{x\in D}\langle h, x - S_0\rangle \le \langle h, T - S_0\rangle,$$

so that $\mathrm{RAROC}(\langle h, S_1 - S_0\rangle) < \lambda_*^{-1}$. The third case is analyzed in a similar way. □

As a corollary, in the case, where $\mathcal{PD} = \{\mathsf{P}\}$ and $H = \mathbb{R}^d$, the solution to the optimization problem is found as follows (see Figure 2). Let T be the intersection of the ray (E, S_0) (in this case $E = \mathsf{E}_\mathsf{P} S_1$) with the border of G. Then

$$\sup_{h\in\mathbb{R}^d} \mathrm{RAROC}(\langle h, S_1 - S_0\rangle) = \frac{|\mathsf{E}_\mathsf{P} S_1 - S_0|}{|S_0 - T|},$$

while $\mathrm{argmax}_{h\in\mathbb{R}^d}\,\mathrm{RAROC}(\langle h, S_1 - S_0\rangle)$ is

$$N_G(T) := \{h \in \mathbb{R}^d : \forall x \in G^\circ,\ \langle h, x - T\rangle > 0\}.$$

In the case, where G has a non-empty interior, $N_G(T)$ is the set of inner normals to G at the point T.

`Important remark`. In order to find the solution of the optimization problem for the case $\mathcal{PD} = \{\mathsf{P}\}$, $H = \mathbb{R}^d$, one needs to know the generating set G and the vector $\mathsf{E}_\mathsf{P} S_1$. The empirical estimation of G is a problem similar to the empirical estimation of volatility, and hence, it can be successfully accomplished. However, the empirical estimation of the mean vector $\mathsf{E}_\mathsf{P} S_1$ is known to be a very unpleasant problem because it is very close to 0 (see the discussion in [4] and the 20's example in [17]). But the well-known security

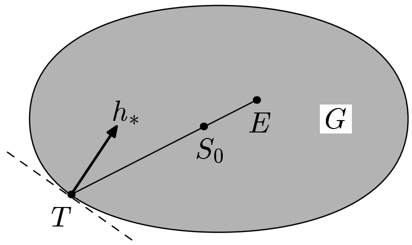

Figure 2. Solution of the optimization problem in the case $\mathcal{PD} = \{\mathsf{P}\}$ and $H = \mathbb{R}^d$

market line relationship of Sharpe [30][4] helps to overcome this problem. This relation states that

$$\mathsf{E}_\mathsf{P}\left(\frac{\widetilde{S}_1^i - \widetilde{S}_0^i}{\widetilde{S}_0^i} - r\right) = \beta^i \mathsf{E}_\mathsf{P}\left(\frac{\widetilde{S}_1^M - \widetilde{S}_0^M}{\widetilde{S}_0^M} - r\right), \quad i = 1, \dots, d,$$

where r is the risk-free interest rate, $\widetilde{S}_n^i = (1+r)^n S_n^i$ are true (not discounted) prices, and $\widetilde{S}_n^M$ is the price of the market portfolio at time n. Hence,

$$\mathsf{E}_\mathsf{P}(S_1^i - S_0^i) = \beta^i const, \quad i = 1, \dots, d.$$

The constant here contains as a factor the expected excess return on the market portfolio, which is again hard to estimate. But note that for our purposes this unknown constant is not needed! Indeed, the geometric solution of the optimization problem presented above requires only the direction of the vector $\mathsf{E}_\mathsf{P} S_1 - S_0$, and this depends only on $(\beta^1, \dots, \beta^d)$.

The following example shows that in natural situations the set of optimal strategies h_* might not be unique (of course, the uniqueness of h_* should be understood up to multiplication by a positive constant).

Example 2.2. Let S_1^1 have a continuous distribution with $\mathsf{E} S_1^1 < \infty$ and take $S_1^2 = (S_1^1 - K)^+$ (so that the second asset is a call option on the first one). Let $\mathcal{PD} = \{\mathsf{P}\}$, $\mathcal{RD}$ be the determining set of Tail V@R of order λ (see [6; Ex. 2.5]) and $H = \mathbb{R}^2$. Assume that $\mathcal{F} = \sigma(S_1^1)$. It is easy to see that $\mathcal{X}_{\mathcal{RD}}(S_1^1)$ consists of a unique element $\mathsf{Q} = \lambda^{-1} I(S_1^1 \le q_\lambda)\mathsf{P}$, where q_λ is the λ-quantile of S_1^1. The border of G has an angle $\pi/4$ at the point $T = \mathsf{E}_\mathsf{Q}(S_1^1, S_1^2)$ (see Figure 3). Let $S_0 = \frac{T + \mathsf{E}_\mathsf{P} S_1}{2}$. Then $N_G(T) = \{h \in \mathbb{R}^2 : h^1 \ge 0,\ h^2 \ge -h^1\}$. □

Let us now find the solution of the optimization problem in the Gaussian case.

Example 2.3. Let S_1 have Gaussian distribution with mean a and covariance matrix C. Let $\mathcal{PD} = \{\mathsf{P}\}$, $H = \mathbb{R}^d$, and $\mathcal{RD}$ be the determining set of a law invariant coherent utility function u that is finite on Gaussian random variables. Assume that S_0 belongs to the relative interior of G and $S_0 \ne a$.

There exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. Let L denote the image of $\mathbb{R}^d$ under the map $x \mapsto Cx$. It is easy to see that

$$G = a + \{C^{1/2}x : \|x\| \le \gamma\} = a + \{y \in L : \langle y, C^{-1}y \rangle \le \gamma^2\}.$$

[4] The whole CAPM and, in particular, the SML relation admit a version based on coherent risk measures instead of variance; see [9].

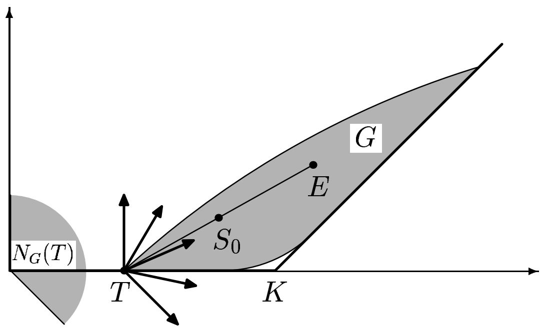

Figure 3. Nonuniqueness of an optimal strategy

We have $T = a + \alpha(S_0 - a)$ with some $\alpha > 0$. It is easy to see that $h \in N_D(T)$ if and only if $\langle h, a - S_0 \rangle > 0$ and, for any $y \in L$ such that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \langle T - a + \varepsilon y, C^{-1}(T - a + \varepsilon y) \rangle = 0,$$

we have $\langle \mathrm{pr}_L h, y \rangle = 0$ (pr_L denotes the orthogonal projection on L). This, in turn, is equivalent to the equality $\mathrm{pr}_L h = c' C^{-1}(a - T) = c C^{-1}(a - S_0)$ with some constant $c > 0$. Thus,

$$N_D(T) = \{h \in \mathbb{R}^d : Ch = c(a - S_0),\ c > 0\}.$$

Note that this set does not depend on u!

It is easy to see that

$$R_* = \frac{|S_0 - a|}{|T - S_0|} = \frac{|S_0 - a|}{|T - a| - |S_0 - a|} = \frac{\langle S_0 - a, C^{-1}(S_0 - a) \rangle^{1/2}}{\gamma - \langle S_0 - a, C^{-1}(S_0 - a) \rangle^{1/2}}.$$

This equality can also be deduced from [6; Ex. 3.14]. □

2.2 Optimal Structure of a Firm

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{D} \subseteq \mathcal{P}$ be a convex set (we assume that $\mathsf{P} \in \mathcal{D}$) and let $X^1, \dots, X^d \in L^1_w(\mathcal{D})$ be the discounted P&Ls produced by different components of some firm (P&L means the Profit&Loss, i.e. the difference between the terminal wealth and the initial wealth).

We will consider the problem

$$\begin{cases} \mathsf{E}_\mathsf{P} \langle h, X \rangle \longrightarrow \max, \\ h \in \mathbb{R}^d_+,\ \rho(\langle h, X \rangle) \le c, \end{cases} \tag{2.4}$$

where c is a positive constant meaning the risk limit on the whole firm. From the financial point of view, (2.4) is the problem of the central management of the firm deciding which components should grow and which should shrink. This is a particular case of the optimization problem of the previous subsection (with $\mathcal{PD} = \{\mathsf{P}\}$, $\mathcal{RD} = \mathcal{D}$, and $H = \mathbb{R}^d_+$), so that we already have a geometric recipe to find the optimal solution. Here we will present an economic characterization of optimality. We will consider an arbitrary convex cone constraint H (not only $\mathbb{R}^d_+$ as in (2.4)). We assume that $\mathsf{E}_\mathsf{P} X \neq 0$ and that the generator G given by (2.1) is strictly convex, i.e. its interior is non-empty and its border contains no interval.

Definition 2.4. We define the *RAROC contribution of X to Y* as

$$\mathrm{RAROC}^c(X;Y) = \frac{\mathsf{E_P}X}{\rho^c(X;Y)},$$

where ρ^c is the risk contribution (see [6; Subsect. 2.5]).

The RAROC contribution is well defined provided that $\rho^c(X;Y)$ is well defined and $\rho^c(X;Y) \neq 0$.

Remarks. (i) The RAROC contribution may take on negative values.
(ii) We have $\mathrm{RAROC}^c(X;X) = \mathrm{RAROC}(X)$.

Theorem 2.5. *If $h \in H$ and*

$$\mathrm{RAROC}^c\Big(h^1X^1; \sum_{i=1}^d h^iX^i\Big) = \dots = \mathrm{RAROC}^c\Big(h^dX^d; \sum_{i=1}^d h^iX^i\Big), \tag{2.5}$$

then $h \in \operatorname{argmax}_{h\in H} \mathrm{RAROC}(\langle h, X\rangle)$ and all the elements of this equality are equal to R_*.
Conversely, if h is an inner point of H and $h \in \operatorname{argmax}_{h\in H} \mathrm{RAROC}(\langle h, X\rangle)$, then (2.5) is satisfied.

Proof. Denote $\sum_i h^iX^i$ by Y. Obviously, $u^c(h^iX^i;Y) = h^iu^c(X^i;Y)$. Repeating the arguments of the proof of [6; Th. 2.12], we get $u^c(X^i;Y) = U^i$, where $U = \operatorname{argmin}_{x\in G}\langle h, x\rangle$, and U is unique due to the strict convexity of G. Thus, (2.5) is equivalent to: $\mathsf{E_P}X = -RU$, where $R = \mathrm{RAROC}^c(h^iX^i;Y)$. It is seen from the geometric results of the previous subsection that this condition implies that $h \in \operatorname{argmax}_{h\in\mathbb{R}^d} \mathrm{RAROC}(\langle h, X\rangle)$. As $u(\langle h, X\rangle) = \langle h, U\rangle$, we get $\mathrm{RAROC}(\langle h, X\rangle) = R$, so that $R_* = R$.

Conversely, if h is an inner point of H, then $\operatorname{argmin}_{x\in D}\langle h, x\rangle = \operatorname{argmin}_{x\in G}\langle h, x\rangle = U$ (D is given by (2.1)). Moreover, $u^c(X^i;Y) = U^i$. Recalling the results of the previous subsection, we see that $U = -R_*\mathsf{E_P}X$, which yields the second statement. □

Remark. The additional assumption that h is in the interior of H is essential for the converse statement of Theorem 2.5. As an example, take $H = \{\alpha h_0 : \alpha \in \mathbb{R}_+\}$, where h_0 is a fixed vector. Then clearly $h_0 \in \operatorname{argmax}_{h\in H} \mathrm{RAROC}(\langle h, X\rangle)$, but of course (2.5) might be violated.

2.3 Single-Agent Optimization

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, $A \subseteq L^0$ be a $\mathcal{D}$-consistent convex cone, and $W \in L^1_s(\mathcal{D})$. From the financial point of view, W is the terminal endowment of some agent, while A is the set of additional discounted P&Ls the agent can obtain by trading.

Problem (single-agent optimization): Find

$$u_* = \sup_{X\in A} u(W+X)$$

and

$$X_* = \operatorname*{argmax}_{X\in A} u(W+X).$$

Proposition 2.6. *We have*

$$u_* = \inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E}_\mathsf{Q} W,$$

where $\mathcal{R}$ *is the set of risk-neutral measures and* $\inf\emptyset := \infty$.

Proof. This statement is, in fact, a reformulation of [6; Prop. 3.20]. □

We will now study the problem for a static model with a finite number of assets. Let $\mathcal{D}\subseteq\mathcal{P}$ be a convex set, $A=\{\langle h,X\rangle : h\in H\}$, where $X=(X^1,\dots,X^d)\in L^1_w(\mathcal{D})$ and $H\subseteq\mathbb{R}^d$ is a closed convex cone. For the case $H=\mathbb{R}^d$, we provided a geometric solution of this problem in [6; Subsect. 3.6]. For an arbitrary H, it is more complicated and is given below. Let us introduce the notation (see Figure 4)

$$\begin{aligned}
G &= \operatorname{cl}\{\mathsf{E}_\mathsf{Q}(X,W) : \mathsf{Q}\in\mathcal{D}\},\\
\widetilde{H} &= \{x\in\mathbb{R}^{d+1} : (x^1,\dots,x^d)\in H,\ x^{d+1}=1\},\\
\widetilde{H}^* &= \{x\in\mathbb{R}^{d+1} : \forall h\in\widetilde{H},\ \langle h,x\rangle\le 0\},\\
e &= (0,\dots,0,1),\\
\lambda_* &= \inf\{\lambda\in\mathbb{R} : (\lambda e+\widetilde{H}^*)\cap G\ne\emptyset\},\\
\widetilde{N} &= \{h\in\mathbb{R}^{d+1} : h^{d+1}=1 \text{ and } \exists a\in\mathbb{R} :\\
&\qquad \forall x\in\lambda_* e+\widetilde{H}^*,\ \forall y\in G,\ \langle h,x\rangle\le a\le\langle h,y\rangle\},\\
N &= \{h\in\mathbb{R}^d : (h,1)\in\widetilde{N}\}.
\end{aligned}$$

If $\lambda_*=\infty$, we set $\widetilde{N}=N=\emptyset$.

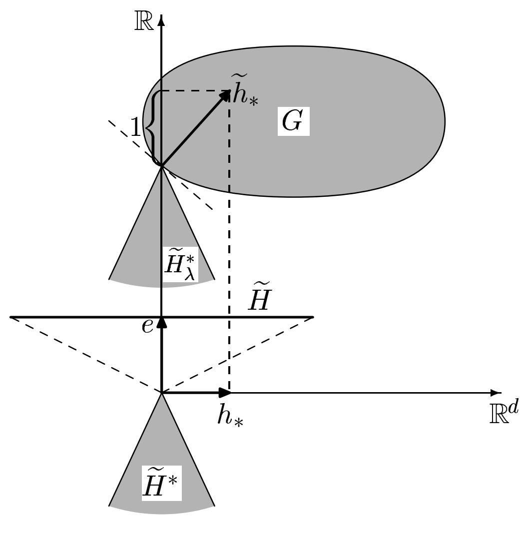

Figure 4. Solution of the optimization problem. By $\widetilde{H}^*_\lambda$ we denote $\lambda_* e+\widetilde{H}^*$. Here $\widetilde{N}=\{\widetilde{h}_*\}$ and $N=\{h_*\}$.

Theorem 2.7. *We have* $u_*=\lambda_*$ *and* $\operatorname{argmax}_{h\in H} u(W+\langle h,X\rangle)=N$.

Proof. Fix $\lambda < \lambda_*$. As G is a convex compact and $\widetilde{H}^*$ is convex and closed, there exist $\widetilde{h} \in \mathbb{R}^{d+1}$ and $a, b \in \mathbb{R}$ such that, for any $x \in \lambda e + \widetilde{H}^*$ and any $y \in G$, we have $\langle \widetilde{h}, x \rangle \le a < b \le \langle \widetilde{h}, y \rangle$. As G is compact, $\widetilde{h}$ can be chosen in such a way that $\widetilde{h}^{d+1} \neq 0$. Since $\widetilde{H}^* \supseteq \{\alpha e : \alpha \le 0\}$, we have $\widetilde{h}^{d+1} > 0$. Without loss of generality, $\widetilde{h}^{d+1} = 1$. Then, for any $x \in \widetilde{H}^*$, we have $\langle \widetilde{h}, x \rangle \le a - \lambda$. As $\widetilde{H}^*$ is a cone, for any $x \in \widetilde{H}^*$, we have $\langle \widetilde{h}, x \rangle \le 0$ and $a - \lambda \ge 0$. Let h be the d-dimensional vector that consists of the first d components of $\widetilde{h}$. Assume that $h \notin H$. Then in the d-dimensional plane $\{x \in \mathbb{R}^{d+1} : x^{d+1} = 1\}$ we can select a $(d-1)$-dimensional plane L that separates $\widetilde{h}$ from $\widetilde{H}$. Consider the d-dimensional plane generated by the origin of $\mathbb{R}^d$ and L, and let x be its normal such that $\langle \widetilde{h}, x \rangle > 0$. Then $\sup_{g \in \widetilde{H}} \langle g, x \rangle \le 0$. Consequently, $x \in \widetilde{H}^*$, but then we get a contradiction with the choice of $\widetilde{h}$. As a result, $h \in H$. Furthermore,

$$u(W + \langle h, X \rangle) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}}(W + \langle h, X \rangle) = \inf_{x \in G} \langle \widetilde{h}, x \rangle > \lambda.$$

As $\lambda < \lambda_*$ has been chosen arbitrarily, we conclude that $\sup_{h \in H} u(W + \langle h, X \rangle) \ge \lambda_*$.

Let us prove the reverse inequality. We can assume that $\lambda_* < \infty$. Let $x_0 \in (\lambda_* e + \widetilde{H}^*) \cap G$. Fix $h \in H$ and set $\widetilde{h} = (h, 1)$. Then

$$u(W + \langle h, X \rangle) = \inf_{x \in G} \langle \widetilde{h}, x \rangle \le \langle \widetilde{h}, x_0 \rangle.$$

We can write $x_0 = \lambda_* e + z_0$ with $z_0 \in \widetilde{H}^*$. Then $\langle \widetilde{h}, x_0 \rangle = \lambda_* + \langle \widetilde{h}, z_0 \rangle \le \lambda_*$. Thus, $\sup_{h \in H} u(W + \langle h, X \rangle) \le \lambda_*$. As a result, $u_* = \lambda_*$.

Let us prove the equality $\operatorname{argmax}_{h \in H} u(W + \langle h, X \rangle) = N$. In the case $\lambda_* = \infty$, its left-hand side and its right-hand side are empty, so it is trivially satisfied. Assume now that $\lambda_* < \infty$. Let $h \in N$. Using the same arguments as above, we show that $h \in H$. For $\widetilde{h} = (h, 1)$, there exists $a \in \mathbb{R}$ such that, for any $x \in \lambda_* e + \widetilde{H}^*$ and any $y \in G$, we have $\langle h, x \rangle \le a \le \langle h, y \rangle$. The same arguments as above show that $a \ge \lambda_*$. Consequently,

$$u(W + \langle h, X \rangle) = \inf_{x \in G} \langle \widetilde{h}, x \rangle \ge a \ge \lambda_*.$$

Let $h \in H$ be such that $u(W + \langle h, X \rangle) = \lambda_*$. This means that, for $\widetilde{h} = (h, 1)$, we have $\inf_{x \in G} \langle \widetilde{h}, x \rangle \ge \lambda_*$. Furthermore, for any $x = \lambda_* e + z \in \lambda_* e + \widetilde{H}^*$, we have $\langle \widetilde{h}, x \rangle = \langle \widetilde{h}, \lambda_* e \rangle + \langle \widetilde{h}, z \rangle \le \lambda_*$. Thus, $\widetilde{h} \in \widetilde{N}$, which means that $h \in N$. □

Example 2.8. (i) Let $H = \mathbb{R}^d$. Then $\widetilde{H}^* = \{\alpha e : \alpha \le 0\}$, and $\lambda_* = \inf\{x^{d+1} : x \in G_0\}$, where $G_0 = G \cap (\{0\} \times \mathbb{R})$. The condition $u_* < \infty$ is equivalent to: $G_0 \neq \emptyset$. If $G^\circ \cap (\{0\} \times \mathbb{R}) \neq \emptyset$, where G° denotes the relative interior of G, then $N \neq \emptyset$. If $G^\circ \cap (\{0\} \times \mathbb{R}) = \emptyset$, then both cases $N \neq \emptyset$ and $N = \emptyset$ are possible (see Figure 5).

(ii) Let $H = \mathbb{R}^d_+$. Then $\widetilde{H}^* = \mathbb{R}^{d+1}_-$, and $\lambda_* = \inf\{x^{d+1} : x \in G_-\}$, where $G_- = G \cap (\mathbb{R}^d_- \times \mathbb{R})$. □

2.4 Liquidity Effects in the NGD Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, and $A \subseteq L^0$ be a convex $\mathcal{D}$-adapted set containing zero. We assume that there exists no $X \in A$ with $u(X) > 0$, i.e. the utility-based NGD condition is satisfied. Let F be the discounted payoff of a contingent claim.

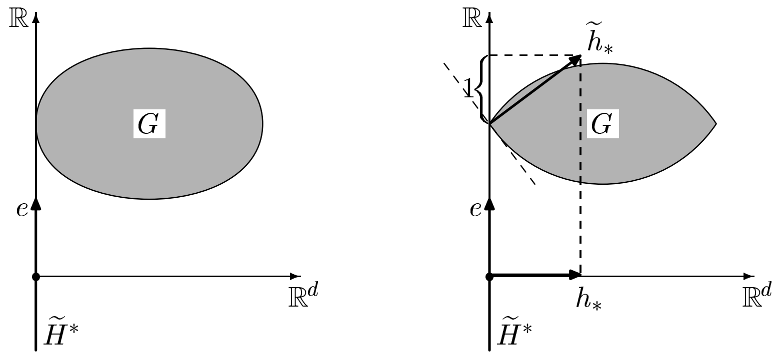

Figure 5. Existence (right) and nonexistence (left) of an optimal strategy for the case $H = \mathbb{R}^d$

Definition 2.9. We define the *upper* and *lower utility-based NGD price functions* of F as

$$\overline{V}(F, v) = \sup\{x : \text{the model } (\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A - v(F - x)) \text{ satisfies the NGD}\}, \quad v > 0,$$
$$\underline{V}(F, v) = \inf\{x : \text{the model } (\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A + v(F - x)) \text{ satisfies the NGD}\}, \quad v > 0.$$

From the financial point of view, v means the volume of a trade.

`Remark`. If A is a cone, then

$$\overline{V}(F, \cdot\,) \equiv \inf\{x : \exists X \in A : u(X - F + x) \geq 0\},$$
$$\underline{V}(F, \cdot\,) \equiv \sup\{x : \exists X \in A : u(X + F - x) \geq 0\}.$$

These are the upper and the lower prices, which were studied in [6; Subsect. 3.6]. Thus, the investigation of $\overline{V}(F, v)$ and $\underline{V}(F, v)$ is meaningful only if A does not have a cone structure. This corresponds to the liquidity effects.

In view of the equality $\underline{V}(F, v) = -\overline{V}(-F, v)$, it is sufficient to study only the properties of $\overline{V}(F, \cdot\,)$.

Theorem 2.10. *Let* $F \in L^1_s(\mathcal{D})$.
(i) *The function* $\overline{V}(F, \cdot\,)$ *is increasing and continuous.*
(ii) *We have*

$$\lim_{v \downarrow 0} \overline{V}(F, v) = \sup_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F.$$

(iii) *We have*

$$\lim_{v \to \infty} \overline{V}(F, v) \leq \sup_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}} F.$$

If $\sup_{X \in A, \, \mathsf{Q} \in \mathcal{D}} |\mathsf{E}_{\mathsf{Q}} X| < \infty$, *then*

$$\lim_{v \to \infty} \overline{V}(F, v) = \sup_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}} F.$$

`Proof`. **(i)** It follows from the equality

$$\sup_{X \in A} u(-v(F - x) + X) = vx + \sup_{X \in A} u(-vF + X)$$

that $\overline{V}(F,v) = -v^{-1}f(v)$, where $f(v) = \sup_{X\in A} u(-vF+X)$. Note that f is finite due to the NGD and the condition $F \in L_s^1(\mathcal{D})$. Fix $v_1, v_2 > 0$, $\varepsilon > 0$, $\alpha \in [0,1]$ and find $X_1, X_2 \in A$ such that $u(-v_iF + X_i) \ge f(v_i) - \varepsilon$, $i = 1,2$. Then

$$\begin{aligned} f(\alpha v_1 + (1-\alpha)v_2) &\ge u(-(\alpha v_1 + (1-\alpha)v_2)F + \alpha X_1 + (1-\alpha)X_2)\\ &\ge \alpha u(-v_1F + X_1) + (1-\alpha)u(-v_2F + X_2)\\ &\ge \alpha f(v_1) + (1-\alpha)f(v_2) - \varepsilon. \end{aligned}$$

Consequently, f is concave. As A contains zero and the NGD is satisfied, we have $f(0) = 0$. This leads to the desired statement.

(ii) By Proposition 2.6,

$$\sup_{X\in \mathrm{cone}\, A} u(-vF+X) = \inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E_Q}(-vF) = -v \sup_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E_Q}F,$$

where "cone" denotes the cone hull. Take $\varepsilon > 0$ and find $X_0 \in A$, $\alpha_0 \ge 0$ such that

$$u(-F + \alpha_0 X_0) \ge -\sup_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E_Q}F - \varepsilon.$$

As the function $\mathbb{R}_+ \ni x \mapsto u(-xF + x\alpha_0X_0)$ is concave and vanishes at zero, we have

$$u(-vF + v\alpha_0X_0) \ge v\Big(-\sup_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E_Q}F - \varepsilon\Big), \quad v \le 1.$$

As $\varepsilon > 0$ has been chosen arbitrarily, we get

$$\limsup_{v\downarrow 0} \overline{V}(F,v) = \limsup_{v\downarrow 0}(-v^{-1}f(v)) \le \sup_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E_Q}F.$$

Combining this with the inequality

$$\sup_{X\in A} u(-vF+X) \le \sup_{X\in A} \inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E_Q}(-vF+X) = \inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E_Q}(-vF) = -v \sup_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E_Q}F,$$

we get the desired statement.

(iii) The first statement follows from the inequality

$$\sup_{X\in A} u(-vF+X) \ge u(-vF) = -v \sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E_Q}F.$$

The second statement is an obvious consequence of the equality $\overline{V}(F,v) = -\sup_{X\in A} u(-F + v^{-1}X)$. □

`Remark`. If $\sup_{X\in A,\, \mathsf{Q}\in\mathcal{D}} |\mathsf{E_Q}X| < \infty$, then

$$\overline{V}(F,\infty) - \underline{V}(F,\infty) = \sup_{\mathsf{Q}\in\mathcal{D}} F - \inf_{\mathsf{Q}\in\mathcal{D}} F,$$

which is the length of the NGD price interval in the absence of a market. The difference

$$\overline{V}(F,0) - \underline{V}(F,0) = \sup_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} F - \inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} F$$

is the length of the NGD price interval in the presence of a market. Thus, the ratio

$$\frac{\overline{V}(F,0) - \underline{V}(F,0)}{\overline{V}(F,\infty) - \underline{V}(F,\infty)}$$

measures the "closeness" of a new instrument F to those already existing in the market.

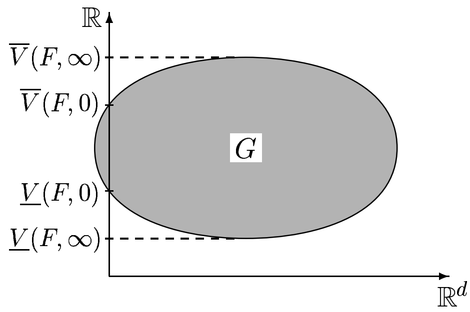

Figure 6. The form of $\overline{V}(F,0)$, $\overline{V}(F,\infty)$, $\underline{V}(F,0)$, and $\underline{V}(F,\infty)$

Example 2.11. Consider a static model with a finite number of assets, i.e. $A = \{\langle h, X\rangle : h \in H\}$, where $X = (X^1,\dots,X^d) \in L^1_w(\mathcal{D})$ and $H \subseteq \mathbb{R}^d$ is a convex bounded set. Assume that H contains a neighborhood of zero. Consider the generator $G = \{\mathsf{E_Q}(X,F) : \mathsf{Q} \in \mathcal{D}\}$. Then

$$\overline{V}(F,0) = \sup\{x^{d+1} : x^1 = \dots = x^d = 0,\ x \in G\},$$
$$\overline{V}(F,\infty) = \sup\{x^{d+1} : x \in G\}$$

(see Figure 6). Note that these values do not depend on H! □

3 Optimality Pricing

3.1 Agent-Independent Optimality Pricing

Consider the model of [6; Subsect. 3.2]. Thus, we are given a probability space $(\Omega,\mathcal{F},\mathsf{P})$, a convex weakly compact set $\mathcal{RD} \subseteq \mathcal{P}$, an L^1-closed convex set $\mathcal{PD} \subseteq \mathcal{RD}$, and a convex $\mathcal{RD}$-adapted set $A \subseteq L^0$. Assume that $0 < R_* < \infty$, where $R_* = \sup_{X\in A} \mathrm{RAROC}(X)$. It follows from [6; Cor. 3.10] that

$$R_* = \inf\Big\{R \ge 0 : \Big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Big) \cap \mathcal{R} \neq \emptyset\Big\}$$

and $\mathcal{D}_* \cap \mathcal{R} \neq \emptyset$, where

$$\mathcal{D}_* = \frac{1}{1+R_*}\mathcal{PD} + \frac{R_*}{1+R_*}\mathcal{RD}.$$

Definition 3.1. An *agent-independent NBC price* of a contingent claim F is a real number x such that

$$\sup_{X\in A+A(x)} \mathrm{RAROC}(X) = \sup_{X\in A} \mathrm{RAROC}(X),$$

where $A(x) = \{h(F-x) : h \in \mathbb{R}\}$. The set of the NBC prices will be denoted by $I_{NBC}(F)$.

This pricing technique corresponds to the agent-independent optimization. A price x is fair if adding to the market a new instrument with the initial price x and the terminal price F does not increase the optimal value of the RAROC in this optimization problem.

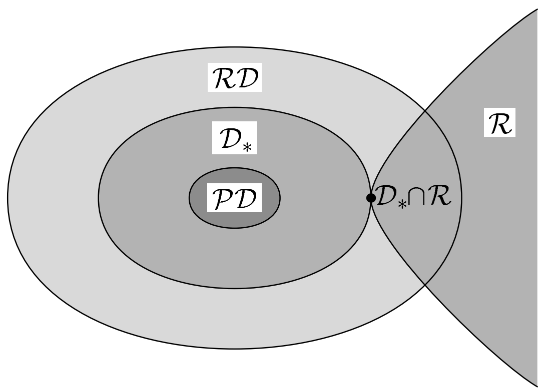

Figure 7. The structure of $\mathcal{D}_* \cap \mathcal{R}$

Proposition 3.2. *For* $F \in L^1_s(\mathcal{RD})$,

$$I_{NBC}(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{D}_* \cap \mathcal{R}\}.$$

`Proof`. If $x \in I_{NBC}(F)$, then, by [6; Cor. 3.10], there exists $\mathsf{Q} \in \mathcal{D}_* \cap \mathcal{R}(A + A(x))$. This means that $\mathsf{Q} \in \mathcal{D}_* \cap \mathcal{R}$ and $\mathsf{E}_\mathsf{Q} F = x$.

Conversely, if $x = \mathsf{E}_\mathsf{Q} F$ with some $\mathsf{Q} \in \mathcal{D}_* \cap \mathcal{R}$, then, for any $X + h(F - x) \in A + A(x)$, we have $\mathsf{E}_\mathsf{Q} X \le 0$, so that $\mathsf{Q} \in \mathcal{R}(A + A(x))$. Due to [6; Cor. 3.10], $\sup_{X \in A + A(x)} \mathrm{RAROC}(X) \le R_*$. □

The following statement yields a more definite representation of $\mathcal{D}_* \cap \mathcal{R}$.

Proposition 3.3. *If* $\mathcal{PD} = \{\mathsf{P}\}$, $X_* \in \operatorname{argmax}_{X \in A} \mathrm{RAROC}(X)$, *and* $\mathcal{X}_{\mathcal{RD}}(X_*)$ *consists of a unique measure* Q_*, *then*

$$\mathcal{D}_* \cap \mathcal{R} = \left\{ \frac{1}{1 + R_*} \mathsf{P} + \frac{R_*}{1 + R_*} \mathsf{Q}_* \right\}. \tag{3.1}$$

`Proof`. Take an arbitrary measure

$$\mathsf{Q}_0 = \frac{1}{1 + R_*} \mathsf{P} + \frac{R_*}{1 + R_*} \mathsf{Q}_1 \in \mathcal{D}_* \cap \mathcal{R}.$$

We have

$$\mathsf{E}_\mathsf{P} X_* + R_* \inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_\mathsf{Q} X_* \le \mathsf{E}_\mathsf{P} X_* + R_* \mathsf{E}_{\mathsf{Q}_1} X_* \le \mathsf{E}_{\mathsf{Q}_0} X_* \le 0$$

(the second inequality follows from our generalized definition of the expectation, while the third inequality follows from the inclusion $\mathsf{Q}_0 \in \mathcal{R}$). In view of the equality

$$\mathrm{RAROC}(X_*) = \frac{\mathsf{E}_\mathsf{P} X}{- \inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_\mathsf{Q} X} = R_*,$$

we get

$$\mathsf{E}_\mathsf{P} X_* + R_* \inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_\mathsf{Q} X = \mathsf{E}_\mathsf{P} X_* + R_* \mathsf{E}_{\mathsf{Q}_1} X_*.$$

It follows that $\mathsf{Q}_1 \in \mathcal{X}_\mathcal{D}(X_*)$, i.e.

$$\mathcal{D}_* \cap \mathcal{R} \subseteq \left\{ \frac{1}{1 + R_*} \mathsf{P} + \frac{R_*}{1 + R_*} \mathsf{Q}_* \right\}.$$

As the set on the left-hand side is non-empty and the set on the right-hand side is a singleton, we get the desired equality. □

Remarks. (i) If $\mathcal{RD}$ is the determining set of Weighted V@R (see [6; Ex. 2.5]) and X_* has a continuous distribution, then $\mathcal{X}_{\mathcal{RD}}(X_*)$ is a singleton (see [7; Sect. 6]). Thus, in the most natural cases I_{NBC} consists of one point.

(ii) One of techniques for pricing in incomplete markets consists in finding the representative of the set of risk-neutral measures that is the closest one to P in some sense (typically the relative entropy or some other measure of distance is minimized). Note that the set $\mathcal{D}_* \cap \mathcal{R}$ is exactly the set of measures Q from $\mathcal{R}$ that are the closest ones to $\mathcal{PD}$, the "distance" being measured by

$$\rho(\mathsf{Q}, \mathcal{RD}) = \inf\left\{R : \exists \mathsf{Q}_1 \in \mathcal{PD},\ \mathsf{Q}_2 \in \mathcal{RD} : \frac{1}{1+R}\mathsf{Q}_1 + \frac{R}{1+R}\mathsf{Q}_2 = \mathsf{Q}\right\}.$$

We will now study the problem for a static model with a finite number of assets. Let $\mathcal{PD} \subseteq \mathcal{RD} \subseteq \mathcal{P}$ be convex sets, $A = \{\langle h, S_1 - S_0\rangle : h \in H\}$, where $S_0 \in \mathbb{R}^d$, $S_1^1, \dots, S_1^d \in L_w^1(\mathcal{RD})$, and $H \subseteq \mathbb{R}^d$ is a closed convex cone. Assume that $0 < R_* < \infty$. Let $F \in L_w^1(\mathcal{D})$ be a contingent claim. Let us introduce the notation (see Figure 8)

$$\begin{aligned}
H^* &= \{x \in \mathbb{R}^d : \forall h \in H,\ \langle h, x\rangle \ge 0\},\\
\widetilde{H}^* &= H^* \times \{0\},\\
\widetilde{E} &= \mathrm{cl}\{\mathsf{E}_\mathsf{Q}(S_1, F) : \mathsf{Q} \in \mathcal{PD}\},\\
\widetilde{G} &= \mathrm{cl}\{\mathsf{E}_\mathsf{Q}(S_1, F) : \mathsf{Q} \in \mathcal{RD}\},\\
\widetilde{D} &= \widetilde{G} + \widetilde{H}^*,\\
\widetilde{D}_R &= \frac{1}{1+R}\widetilde{E} + \frac{R}{1+R}\widetilde{D}.
\end{aligned}$$

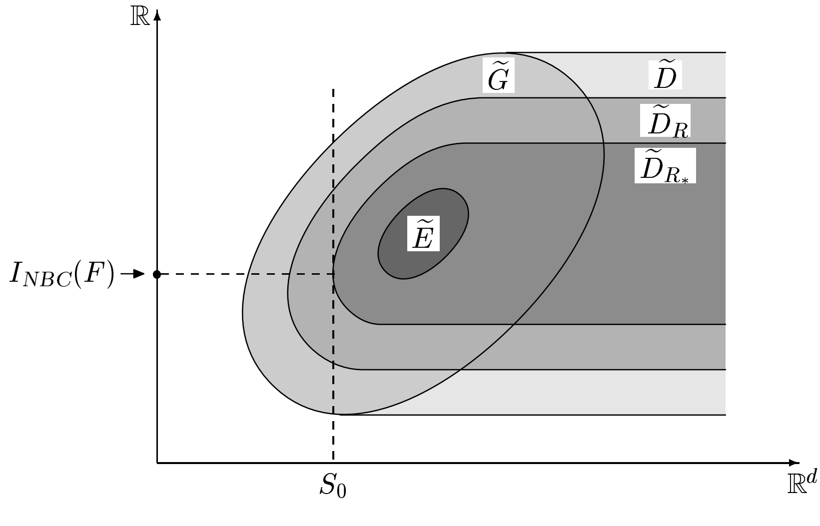

Figure 8. The form of I_{NBC}. Here $I_{NBC}(F)$ consists of one point.

Theorem 3.4. *We have*

$$R_* = \inf\{R > 0 : \widetilde{D}_R \cap (\{S_0\} \times \mathbb{R}) \neq \emptyset\}, \tag{3.2}$$

$$I_{NBC}(F) = \{x : (S_0, x) \in \widetilde{D}_{R_*}\}. \tag{3.3}$$

`Proof`. Denote

$$\begin{aligned} E &= \mathrm{cl}\{\mathsf{E}_\mathsf{Q} S_1 : \mathsf{Q} \in \mathcal{PD}\}, \\ G &= \mathrm{cl}\{\mathsf{E}_\mathsf{Q} S_1 : \mathsf{Q} \in \mathcal{RD}\}, \\ D &= G + H^*, \\ D_R &= \frac{1}{1+R}\, E + \frac{R}{1+R}\, D. \end{aligned}$$

Note that $E = \mathrm{pr}_{\mathbb{R}^d} \widetilde{E}$, $G = \mathrm{pr}_{\mathbb{R}^d} \widetilde{G}$, $H^* = \mathrm{pr}_{\mathbb{R}^d} \widetilde{H}^*$, and consequently, $D = \mathrm{pr}_{\mathbb{R}^d} \widetilde{D}$, $D_R = \mathrm{pr}_{\mathbb{R}^d} \widetilde{D}_R$. Combining this with the results of Subsection 2.1, we get

$$R_* = \inf\{R > 0 : D_R \ni S_0\} = \inf\{R > 0 : \widetilde{D}_R \cap (\{S_0\} \times \mathbb{R}) \neq \emptyset\}.$$

Furthermore, for any $x \in \mathbb{R}$,

$$\sup_{A + A(x)} \mathrm{RAROC}(X) = \inf\{R > 0 : \widetilde{D}_R \ni (S_0, x)\}.$$

This, combined with (3.2), proves (3.3). □

To conclude this subsection, we find the form of $I_{NBC}(F)$ in the Gaussian case.

Example 3.5. Consider the setting of [6; Ex. 3.14]. Clearly, R_* is the solution of the equation $\langle S_0 - a, C^{-1}(S_0 - a)\rangle = \frac{\gamma^2 R_*^2}{(1+R_*)^2}$ (cf. Example 2.3). This, combined with the form of $I_{NGD\,(R)}(F)$ found in [6; Ex. 3.14], shows that $I_{NBC}(F)$ consists of a unique point $\langle b, S_0 - a\rangle + \mathsf{E}F$. Let us remark that this value coincides with the fair price of F obtained as a result of the mean-variance hedging. Note that this value does not depend on u! □

3.2 Single-Agent Optimality Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, $A \subseteq L^0$ be a $\mathcal{D}$-consistent convex set containing zero, and $W \in L^1_s(\mathcal{D})$. The financial interpretation is the same as in Subsection 2.3.

Definition 3.6. A *single-agent NBC price* of a contingent claim F is a real number x such that

$$\max_{X \in A,\, h \in \mathbb{R}} u(W + X + h(F - x)) = u(W).$$

The set of the NBC prices will be denoted by $I_{NBC}(F)$.

This pricing technique corresponds to the single-agent optimization. A price x is fair if adding to the market a new instrument with the initial price x and the terminal price F does not increase the optimal value of the utility in this optimization problem.

Theorem 3.7. *For $F \in L^1_s(\mathcal{D})$,*

$$I_{NBC}(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{X}_\mathcal{D}(W) \cap \mathcal{R}\}. \tag{3.4}$$

Remarks. (i) The set of the NBC prices is non-empty only if W is optimal in the sense that $\max_{X\in A} u(W+X)=u(W)$. However, if W is not optimal, then, as seen from the proof of Theorem 3.7, $\mathcal{X}_{\mathcal{D}}(W)\cap\mathcal{R}=\emptyset$, so that (3.4) still holds.

(ii) If W is optimal and $\mathcal{X}_{\mathcal{D}}(W)$ consists of a unique measure Q_*, then $I_{NBC}(F)=\{\mathsf{E}_{\mathsf{Q}_*}F\}$. For example, this is true if $\mathcal{D}$ is the determining set of Weighted V@R (see [6; Ex. 2.5]) and W has a continuous distribution (see [7; Sect. 6]). The dependence on A formally disappears in this case.

Proof of Theorem 3.7. As A contains zero and the function $\mathbb{R}_+\ni\alpha\mapsto u(W+\alpha X)$ is concave for a fixed X, the condition $x\in I_{NBC}(F)$ is equivalent to:

$$\max_{X\in\operatorname{cone} A,\, h\in\mathbb{R}} u(W+X+h(F-x))=u(W).$$

By Proposition 2.6, this is equivalent to:

$$\inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}(A+A(x))}\mathsf{E}_{\mathsf{Q}}W=\inf_{\mathsf{Q}\in\mathcal{D}}\mathsf{E}_{\mathsf{Q}}W,$$

where $A(x)=\{h(F-x):h\in\mathbb{R}\}$. Clearly, the latter condition is equivalent to: $\mathcal{X}_{\mathcal{D}}(W)\cap\mathcal{R}(A+A(x))\neq\emptyset$. It is easy to verify that this is equivalent to: $x=\mathsf{E}_{\mathsf{Q}}F$ for some $\mathsf{Q}\in\mathcal{X}_{\mathcal{D}}(W)\cap\mathcal{R}$. □

Let us now provide a geometric representation of $I_{NBC}(F)$ (see Figure 9). Assume that $u(W)=\max_{X\in A}u(W+X)$ (the reasoning used above shows that this is equivalent to: $\mathcal{X}_{\mathcal{D}}(W)\cap\mathcal{R}\neq\emptyset$). Consider the generator $G=\{\mathsf{E}_{\mathsf{Q}}(F,W):\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}\}$ and the function $f(x)=\inf\{y:(x,y)\in G\}$ (we set $\inf\emptyset=+\infty$).

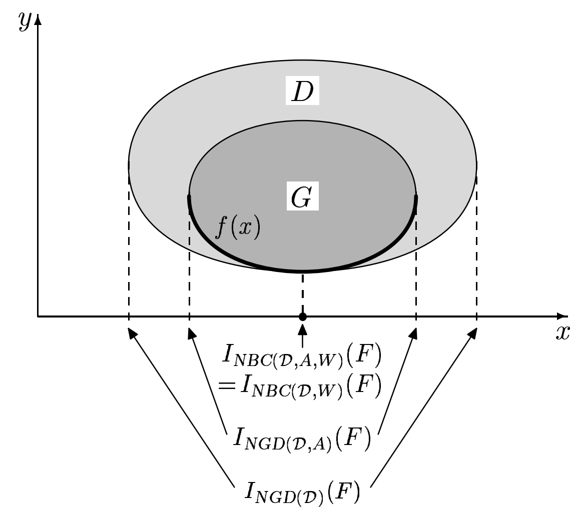

Figure 9. Comparison of various price intervals. Here $G=\{\mathsf{E}_{\mathsf{Q}}(F,W):\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}\}$ and $D=\{\mathsf{E}_{\mathsf{Q}}(F,W):\mathsf{Q}\in\mathcal{D}\}$. In this example, $I_{NBC(\mathcal{D},A,W)}(F)=I_{NBC(\mathcal{D},W)}(F)$.

Corollary 3.8. *For $F\in L^1_s(\mathcal{D})$,*

$$I_{NBC}(F)=\operatorname*{argmin}_{x\in\mathbb{R}} f(x).$$

`Proof`. It is sufficient to note that

$$\min_{x\in\mathbb{R}} f(x) = \min_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E}_{\mathsf{Q}} W = u(W)$$

and

$$f(x) = \inf\{\mathsf{E}_{\mathsf{Q}} W : \mathsf{Q} \in \mathcal{D}\cap\mathcal{R}(A + A(x))\},$$

where $A(x) = \{h(F - x) : h \in \mathbb{R}\}$. Thus, $x \in \operatorname{argmin}_{x\in\mathbb{R}} f(x)$ if and only if $\mathcal{X}_{\mathcal{D}}(W) \cap \mathcal{R}(A + A(x)) \neq \emptyset$, which, in view of Theorem 3.7, is equivalent to the inclusion $x \in I_{NBC}(F)$. □

Assume that W is optimal in the sense that

$$u(W) = \max_{X\in A} u(W + X) \tag{3.5}$$

and suppose moreover that the set $I_{NBC(\mathcal{D},W)}(F)$ of the NBC prices based on $\mathcal{D}$ and W (with $A = 0$) consists of one point x_0 (this condition is satisfied if the set $D = \{\mathsf{E}_{\mathsf{Q}}(F, W) : \mathsf{Q} \in \mathcal{D}\}$ is strictly convex; see Figure 9). It is seen from the proof of Theorem 3.7 that condition (3.5) is equivalent to: $\mathcal{X}_{\mathcal{D}}(W) \cap \mathcal{R} \neq \emptyset$. Then it follows from Theorem 3.7 that $I_{NBC(\mathcal{D},A,W)}(F) \neq \emptyset$ (we assume that $F \in L^1_s(\mathcal{D})$). Clearly, $I_{NBC(\mathcal{D},A,W)}(F) \subseteq I_{NBC(\mathcal{D},W)}(F)$. As a result, $I_{NBC(\mathcal{D},A,W)}(F) = \{x_0\}$. So, in this situation A can be eliminated. This situation occurs naturally as shown, in particular, by the example below.

Example 3.9. Let u be a law invariant coherent utility function that is finite on Gaussian random variables. Assume that $u(W) = \max_{X\in A} u(W + X)$ and that (W, F) has a Gaussian distribution.

There exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. Clearly, $I_{NBC}(F) \subseteq J$, where J is the NBC price based on $\mathcal{D}$ and W with $A = 0$. Using Corollary 3.8, we deduce that J consists of a single point $\mathsf{E}F - \gamma\frac{\mathsf{cov}(F,W)}{(\mathsf{var}\, W)^{1/2}}$. As $I_{NBC}(F)$ is non-empty, it consists of the same point. □

3.3 Multi-Agent Optimality Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $u_1, \dots, u_N$ be coherent utility functions with the weakly compact determining sets $\mathcal{D}_1, \dots, \mathcal{D}_N$, $A \subseteq L^0$ be a convex set containing zero, and $W_1 \in L^1_s(\mathcal{D}_1), \dots, W_N \in L^1_s(\mathcal{D}_N)$. From the financial point of view, u_n, A, and W_n are the coherent utility function, the set of attainable P&Ls, and the terminal endowment of the n-th agent, respectively. We will assume that there exists a set $A' \subseteq \bigcap_n L^1_s(\mathcal{D}_n) \cap A$ such that, for any n, $\mathcal{D}_n \cap \mathcal{R} = \mathcal{D}_n \cap \mathcal{R}(A')$. We also assume that each W_n is optimal in the sense that $u_n(W_n) = \max_{X\in A} u_n(W_n + X)$.

Definition 3.10. A *multi-agent NBC price* of a contingent claim F is a real number x such that there exists no element $X \in A + \{h(F - x) : h \in \mathbb{R}\}$ such that $u_n(W_n + X) > u_n(W_n)$ for any n. The set of the NBC prices will be denoted by $I_{NBC}(F)$.

From the financial point of view, a price x is fair if adding to the market a new instrument with the initial price x and the terminal price F does not produce a trading opportunity that is attractive to all the agents.

Theorem 3.11. *For $F \in \bigcap_n L^1_s(\mathcal{D}_n)$,*

$$I_{NBC}(F) = \operatorname{conv}_{n=1}^N I_{NBC(\mathcal{D}_n,A,W_n)}(F) = \bigl\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \operatorname{conv}_{n=1}^N(\mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R})\bigr\},$$

where $I_{NBC(\mathcal{D}_n,A,W_n)}(F)$ is the interval of the single-agent NBC prices based on $\mathcal{D}_n$, A, W_n.

Proof. Let $x \in I_{NBC}(F)$. Fix $X_1,\dots,X_M \in A'$. It follows from the weak continuity of the maps $\mathcal{D}_n \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q}(X_1,\dots,X_M,F)$ that, for each $n = 1,\dots,N$, the set $G_n = \{\mathsf{E}_\mathsf{Q}(X_1,\dots,X_M,F-x) : \mathsf{Q} \in \mathcal{X}_n\}$, where $\mathcal{X}_n = \mathcal{X}_{\mathcal{D}_n}(W_n)$, is compact. Clearly, G_n is convex. Suppose that

$$\bigl(\operatorname{conv}_{n=1}^N G_n\bigr) \cap (\mathbb{R}^M_- \times \{0\}) = \emptyset.$$

Then there exists $h \in \mathbb{R}^{M+1}$ such that $h_1,\dots,h_M \ge 0$ and $\inf_{x \in G_n}\langle h,x\rangle > 0$ for each n. This means that $\inf_{\mathsf{Q}\in\mathcal{X}_n} \mathsf{E}_\mathsf{Q} Y > 0$ for each n, where $Y = h^1X_1 + \dots + h^mX_m + h^{m+1}(F-x)$. Employing [6; Th. 2.16], we conclude that there exists $\varepsilon > 0$ such that $u(W_n + \varepsilon Y) > u(W_n)$ for any n.

The obtained contradiction shows that, for any $X_1,\dots,X_M \in A'$, the set

$$B(X_1,\dots,X_M) = \Biggl\{\alpha_1,\dots,\alpha_N,\mathsf{Q}_1,\dots,\mathsf{Q}_N \in S \times \prod_{n=1}^N \mathcal{X}_n : \sum_{n=1}^N \alpha_n \mathsf{E}_{\mathsf{Q}_n} F = x \text{ and } \forall n = 1,\dots,N,\ \forall m = 1,\dots,M,\ \mathsf{E}_{\mathsf{Q}_n} X_m \le 0\Biggr\},$$

where $S = \bigl\{\alpha_1,\dots,\alpha_N \ge 0 : \sum_{n=1}^N \alpha_n = 1\bigr\}$, is non-empty. As the map $\mathcal{X}_n \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} X$ is weakly continuous for each $X \in L^1_s(\mathcal{D}_n)$, the set $B(X_1,\dots,X_M)$ is closed with respect to the product of weak topologies. Furthermore, any finite intersection of sets of this form is non-empty. Tikhonov's theorem ensures that $S \times \prod_n \mathcal{X}_n$ is compact. Consequently, there exists a collection $\alpha_1,\dots,\alpha_N,\mathsf{Q}_1,\dots,\mathsf{Q}_N$ that belongs to each B of this form. Then $\mathsf{E}_{\mathsf{Q}_n} X \le 0$ for any n and any $X \in A'$, which means that $\mathsf{Q}_n \in \mathcal{X}_n \cap \mathcal{R}$. Thus, the measure $\mathsf{Q} = \sum_n \alpha_n \mathsf{Q}_n$ belongs to $\operatorname{conv}_n(\mathcal{X}_n \cap \mathcal{R})$ and $\mathsf{E}_\mathsf{Q} F = x$.

Now, let $x = \mathsf{E}_\mathsf{Q} F$ with $\mathsf{Q} = \sum_n \alpha_n \mathsf{Q}_n$, $\mathsf{Q}_n \in \mathcal{X}_n \cap \mathcal{R}$. Suppose that there exist $X \in A$, $h \in \mathbb{R}$ such that, for $Y = X + h(F-x)$, we have $u_n(W_n + Y) > u_n(W_n)$ for each n. Due to the concavity of the function $\alpha \mapsto u_n(W_n + \alpha Y)$, we get

$$\begin{aligned} u_n(W_n+Y) - u_n(W_n) &\le \limsup_{\varepsilon\downarrow 0} \varepsilon^{-1}(u_n(W_n+\varepsilon Y) - u_n(W_n)) \\ &\le \limsup_{\varepsilon\downarrow 0} \varepsilon^{-1}\Bigl(\inf_{\mathsf{Q}\in\mathcal{X}_n} \mathsf{E}_\mathsf{Q}(W_n+\varepsilon Y) - u_n(W_n)\Bigr) = \inf_{\mathsf{Q}\in\mathcal{X}_n} \mathsf{E}_\mathsf{Q} Y. \end{aligned}$$

Consequently, $\mathsf{E}_{\mathsf{Q}_n} Y > 0$ for each n, and therefore, $\mathsf{E}_\mathsf{Q} Y > 0$. But, on the other hand, $\mathsf{Q} \in \mathcal{R}$, and therefore, $\mathsf{E}_\mathsf{Q} Y \le \mathsf{E}_\mathsf{Q} h(F-x) = 0$. The contradiction shows that $x \in I_{NBC}(F)$. □

4 Equilibrium

4.1 Unconstrained Equilibrium

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space, $u_1,\dots,u_N$ be coherent utility functions with the weakly compact determining sets $\mathcal{D}_1,\dots,\mathcal{D}_N$, $A_1,\dots,A_N \subseteq L^0$ be convex cones such that A_n is $\mathcal{D}_n$-consistent for each n, and let $W_1 \in L^1_s(\mathcal{D}_1), \dots, W_N \in L^1_s(\mathcal{D}_N)$. From the financial point of view, u_n, A_n, and W_n are the coherent utility function, the "personal" set of attainable P&Ls (thus, different agents are assumed to have different access to the market), and the terminal endowment of the n-th agent, respectively. Let us introduce the notation $\mathcal{D} = \bigcap_n \mathcal{D}_n$. By $\mathcal{R}_n$ we will denote the set of risk-neutral measures corresponding to A_n.

Definition 4.1. The *maximal overall utility* is defined as

$$M = \sup_{\substack{X_n \in A_n \\ Y_n \in L^1_s(\mathcal{D}) : \sum_n Y_n = 0}} \sum_{n=1}^N u_n(W_n + X_n + Y_n),$$

where the sum is understood as $-\infty$ if any of the summands equals $-\infty$.

Proposition 4.2. *We have*

$$M = \inf_{\mathsf{Q} \in \bigcap_n \mathcal{D}_n \cap \mathcal{R}_n} \mathsf{E}_\mathsf{Q} W,$$

where $W := \sum_n W_n$ *and* $\inf \emptyset := \infty$.

Lemma 4.3. *Let* $u_1, \dots, u_N$ *be coherent utility functions with the weakly compact determining sets* $\mathcal{D}_1, \dots, \mathcal{D}_N$. *Then, for any* $X \in L^\infty$,

$$\sup_{X_n \in L^\infty : \sum_n X_n = X} \sum_{n=1}^N u_n(X_n) = \inf_{\mathsf{Q} \in \bigcap_n \mathcal{D}_n} \mathsf{E}_\mathsf{Q} X. \tag{4.1}$$

`Remark.` The left-hand side of (4.1) is called the *convex convolution* or the *sup-convolution* of $u_1, \dots, u_N$ (see [3], [11; Sect. 5.2]). Thus, Lemma 4.3 states that it is a coherent utility function with the determining set $\bigcap_n \mathcal{D}_n$ if $\bigcap_n \mathcal{D}_n \neq \emptyset$ and it is identically equal to $+\infty$ if $\bigcap_n \mathcal{D}_n = \emptyset$.

`Proof of Lemma 4.3.` In the case, where $\bigcap_n \mathcal{D}_n \neq \emptyset$, this statement follows by induction from a result proved in [11; Sect. 5.2].

Assume now that $\bigcap_n \mathcal{D}_n = \emptyset$. Find m such that $\bigcap_{n=1}^m \mathcal{D}_n \neq \emptyset$, while $\bigcap_{n=1}^{m+1} \mathcal{D}_n = \emptyset$. By the Hahn-Banach theorem, there exists $Z \in L^\infty$ such that

$$\sup_{\mathsf{Q} \in \mathcal{D}_{m+1}} \mathsf{E}_\mathsf{Q} Z < 0 < \inf_{\mathsf{Q} \in \bigcap_{n=1}^m \mathcal{D}_n} \mathsf{E}_\mathsf{Q} Z.$$

According to the part of the lemma that has already been proved, there exist $Z_1, \dots, Z_m \in L^\infty$ such that $\sum_{n=1}^m Z_n = Z$ and $\sum_{n=1}^m u_n(Z_n) > 0$. Then $u_1(Z_1) + \dots + u_m(Z_m) + u_{m+1}(-Z) > 0$. Consequently, the left-hand side of (4.1) is identically equal to $+\infty$. □

`Proof of Proposition 4.2.` For any $X_1 \in A_1, \dots, X_N \in A_N, Y_1, \dots, Y_N \in L^1_s(\mathcal{D})$ such that $\sum_n Y_n = 0$, and any $\mathsf{Q} \in \bigcap_n \mathcal{D}_n \cap \mathcal{R}_n$, we have

$$\sum_{n=1}^N u_n(W_n + X_n + Y_n) \le \sum_{n=1}^N \mathsf{E}_\mathsf{Q}(W_n + X_n + Y_n) \le \sum_{n=1}^N \mathsf{E}_\mathsf{Q}(W_n + Y_n) = \mathsf{E}_\mathsf{Q} W$$

(to get the second inequality, we used the inclusions $W_n \in L^1(\mathsf{Q})$, $Y_n \in L^1(\mathsf{Q})$). Consequently,

$$M \le \inf_{\mathsf{Q}\in\bigcap_n \mathcal{D}_n\cap\mathcal{R}_n} \mathsf{E}_\mathsf{Q} W.$$

Let us prove the reverse inequality. Clearly, it is sufficient to prove it for bounded W_n (since arbitrary $W_n \in L^1_s(\mathcal{D}_n)$ can be approximated by bounded ones). Proposition 2.6 and Lemma 4.3 combined together yield

$$\begin{aligned} M &\ge \sup_{\substack{X_n\in A_n\\ Y_n\in L^\infty:\sum_n Y_n=0}} \sum_{n=1}^N u_n(W_n+X_n+Y_n)\\ &= \sup_{Y_n\in L^\infty:\sum_n Y_n=0} \sum_{n=1}^N \inf_{\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}_n} \mathsf{E}_\mathsf{Q}(W_n+Y_n)\\ &= \inf_{\mathsf{Q}\in\bigcap_n\mathcal{D}_n\cap\mathcal{R}_n} \mathsf{E}_\mathsf{Q} W. \end{aligned}$$

The following example shows that the restriction $Y_n \in L^1_s(\mathcal{D})$ in the definition of M is essential for Proposition 4.2 and cannot be eliminated.

Example 4.4. Let $N = 2$, $\mathcal{D}_1 = \mathcal{D}_2 = \{\mathsf{P}\}$, $A_n = \{a\xi_n + Y : a \in \mathbb{R}, Y \in L^\infty, \mathsf{E}_\mathsf{P} Y = 0\}$, where $\mathsf{E}_\mathsf{P}\xi_n^+ = \mathsf{E}_\mathsf{P}\xi_n^- = \infty$, $n = 1, 2$, $\xi_1 + \xi_2 = 1$, and $W_1 = W_2 = 0$. Take $X_n = \xi_n$, $Y_n = 1/2 - \xi_n$, $n = 1, 2$. Then $Y_1 + Y_2 = 0$ and $W_n + X_n + Y_n = 1/2$, so that

$$\sup_{\substack{X_n\in A_n\\ Y_n\in L^0:\sum_n Y_n=0}} \sum_{n=1}^N u_n(W_n+X_n+Y_n) = \infty.$$

On the other hand, $\bigcap_n \mathcal{D}_n \cap \mathcal{R}_n = \{\mathsf{P}\}$ and $\mathsf{E}_\mathsf{P} W = 0$. □

We now introduce two definitions of equilibrium. From the financial point of view, the Pareto-type equilibrium corresponds to the global optimum, while the Arrow-Debreu-type equilibrium corresponds to the competitive optimum.

Definition 4.5. A *Pareto-type equilibrium* is a collection $(X, Y) = (X_1, \dots, X_N, Y_1, \dots, Y_N)$ such that

(a) $X_n \in A_n$;
(b) $Y_n \in L^1_s(\mathcal{D})$, $\sum_n Y_n = 0$;
(c) there do not exist (X', Y') satisfying (a), (b) and such that

$$\begin{aligned} &\forall n,\ u_n(W_n + X'_n + Y'_n) \ge u_n(W_n + X_n + Y_n),\\ &\exists n : u_n(W_n + X'_n + Y'_n) > u_n(W_n + X_n + Y_n). \end{aligned}$$

`Remark.` It is easy to see from the translation invariance property ($u_n(X + m) = u_n(X) + m$) that condition (c) is equivalent to:

(c') $\sum_n u_n(W_n + X_n + Y_n) = M$.

Definition 4.6. An *Arrow-Debreu-type equilibrium* is a collection (X, Y, Q), where $\mathsf{Q} \in \mathcal{P}$, such that

(a) $X_n \in A_n$;
(b) $Y_n \in L^1_s(\mathcal{D})$, $\sum_n Y_n = 0$, $\mathsf{E}_\mathsf{Q} Y_n = 0$ (so that automatically $Y_n \in L^1(\mathsf{Q})$);
(c) for any n,

$$u_n(W_n + X_n + Y_n) = \max_{\substack{\xi \in A_n \\ \eta \in L^1(\mathsf{Q})\,:\,\mathsf{E}_\mathsf{Q}\eta = 0}} u_n(W_n + \xi + \eta).$$

Below the notation $Y' \sim Y$ for random vectors $(Y'_1, \dots, Y'_N)$ and $(Y_1, \dots, Y_N)$ means that there exist $a_1, \dots, a_N \in \mathbb{R}$ such that $\sum_n a_n = 0$ and $Y'_n = Y_n + a_n$. We denote

$$\mathcal{E}(X, Y) = \{\mathsf{Q} \in \mathcal{P} : \exists Y' \sim Y \text{ such that } (X, Y', \mathsf{Q}) \text{ is an Arrow-Debreu-type equilibrium}\}.$$

Theorem 4.7. *Assume that $M < \infty$ (by Proposition 4.2, this is equivalent to: $\bigcap_n \mathcal{D}_n \cap \mathcal{R}_n \neq \emptyset$). Let $X_n \in A_n$, $Y_n \in L^1_s(\mathcal{D})$, $\sum_n Y_n = 0$. The following conditions are equivalent:*

(i) *(X, Y) is a Pareto-type equilibrium;*
(ii) *there exist $\mathsf{Q} \in \mathcal{P}$ and $Y' \sim Y$ such that (X, Y', Q) is an Arrow-Debreu-type equilibrium.*

If these conditions are satisfied, then

$$\mathcal{E}(X, Y) = \mathcal{X}_{\bigcap_n \mathcal{D}_n \cap \mathcal{R}_n}(W).$$

If each A_n is a linear space, then (i), (ii) *are equivalent to:*

(iii) $\bigcap_n \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + Y_n) \cap \mathcal{R}_n \neq \emptyset$.

Moreover, in this case

$$\mathcal{E}(X, Y) = \bigcap_{n=1}^N \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + Y_n) \cap \mathcal{R}_n.$$

`Proof`. *Step 1.* Let us prove the implication (i)$\Rightarrow$(ii). Take $\mathsf{Q} \in \mathcal{X}_{\bigcap_n \mathcal{D}_n \cap \mathcal{R}_n}(W)$ (this set is non-empty due to [6; Prop. 2.9]). Using Proposition 4.2, we can write

$$\sum_{n=1}^N u_n(W_n + X_n + Y_n) \le \sum_{n=1}^N \mathsf{E}_\mathsf{Q}(W_n + Y_n) \le \mathsf{E}_\mathsf{Q} \sum_{n=1}^N (W_n + Y_n) = \mathsf{E}_\mathsf{Q} W = M$$

(note that the expectation operator understood in the sense of [6; Def. 2.3] has the property $\mathsf{E}(\xi + \eta) \ge \mathsf{E}\xi + \mathsf{E}\eta$). Since the left-hand side and the right-hand side of the above inequality coincide, we get $u_n(W_n + X_n + Y_n) = \mathsf{E}_\mathsf{Q}(W_n + Y_n)$ for any n. We can find $Y' \sim Y$ such that $\mathsf{E}_\mathsf{Q} Y'_n = 0$ for any n. Then, for any n, $\xi \in A_n$, and $\eta \in L^1(\mathsf{Q})$ such that $\mathsf{E}_\mathsf{Q}\eta = 0$, we have

$$u_n(W_n + \xi + \eta) \le \mathsf{E}_\mathsf{Q}(W_n + \xi + \eta) \le \mathsf{E}_\mathsf{Q} W_n = u_n(W_n + X_n + Y'_n).$$

Thus, (X, Y', Q) is an Arrow-Debreu-type equilibrium.

Step 2. Let us prove the implication (ii)$\Rightarrow$(i). Suppose that there exist $\widetilde{X}_n \in A_n$ and $\widetilde{Y}_n \in L^1_s(\mathcal{D}_n)$ with $\sum_n \widetilde{Y}_n = 0$ such that

$$\sum_{n=1}^N u_n(W_n + \widetilde{X}_n + \widetilde{Y}_n) > \sum_{n=1}^N u_n(W_n + X_n + Y_n). \tag{4.2}$$

Fix n and suppose that $\mathsf{Q} \notin \mathcal{D}_n \cap \mathcal{R}_n$. It is easy to check that $\mathcal{D}_n \cap \mathcal{R}_n$ is L^1-closed. By the Hahn-Banach theorem, there exists $\eta \in L^\infty$ such that $\mathsf{E}_\mathsf{Q}\eta < \inf_{\mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n} \mathsf{E}_\mathsf{Q}\eta$. According to Proposition 2.6, there exists $\xi \in A_n$ such that $\mathsf{E}_\mathsf{Q}\eta < u_n(\xi + \eta)$. This means that $u_n(\xi + \eta - \mathsf{E}_\mathsf{Q}\eta) > 0$. Then

$$u_n(W_n + \alpha\xi + \alpha(\eta - \mathsf{E}_\mathsf{Q}\eta)) \xrightarrow[\alpha \to \infty]{} \infty. \tag{4.3}$$

On the other hand, in view of Proposition 4.2, the condition $M < \infty$ implies that $\mathcal{D}_n \cap \mathcal{R}_n \neq \emptyset$. Fix $\widetilde{\mathsf{Q}} \in \mathcal{D}_n \cap \mathcal{R}_n$. Then

$$u_n(W_n + \alpha\xi + \alpha(\eta - \mathsf{E}_\mathsf{Q}\eta)) \le u_n(W_n + X_n + Y_n) \le \mathsf{E}_{\widetilde{\mathsf{Q}}}(W_n + Y_n) < \infty,$$

which contradicts (4.3). Thus, $\mathsf{Q} \in \bigcap_n \mathcal{D}_n \cap \mathcal{R}_n$. In particular, $Y_n \in L^1(\mathsf{Q})$, so that we can find $\widetilde{Y}' \sim \widetilde{Y}$ such that $\mathsf{E}_\mathsf{Q}\widetilde{Y}'_n = 0$ for any n. Then

$$\sum_{n=1}^N u_n(W_n + \widetilde{X}_n + \widetilde{Y}_n) = \sum_{n=1}^N u_n(W_n + \widetilde{X}_n + \widetilde{Y}'_n) \le \sum_{n=1}^N u_n(W_n + X_n + Y_n),$$

which contradicts (4.2).

Step 3. It was shown in Step 1 that

$$\mathcal{X}_{\bigcap_n \mathcal{D}_n \cap \mathcal{R}_n}(W) \subseteq \mathcal{E}(X, Y).$$

Let us prove the reverse inclusion. Take $\mathsf{Q} \in \mathcal{E}(X, Y)$ and find $Y' \sim Y$ such that (X, Y', Q) is an Arrow-Debreu-type equilibrium. It was shown in Step 2 that $\mathsf{Q} \in \bigcap_n \mathcal{D}_n \cap \mathcal{R}_n$. Applying Proposition 2.6 to the $\mathcal{D}_n$-consistent convex cone $A = \{\eta \in L^\infty : \mathsf{E}_\mathsf{Q}\eta = 0\}$, we get

$$\sup_{\eta \in L^1(\mathsf{Q}) : \mathsf{E}_\mathsf{Q}\eta = 0} u_n(W_n + \eta) = \mathsf{E}_\mathsf{Q} W_n, \quad n = 1, \dots, N. \tag{4.4}$$

Thus,

$$M \ge \sum_{n=1}^N u_n(W_n + X_n + Y_n) = \sum_{n=1}^N u_n(W_n + X_n + Y'_n) \ge \sum_{n=1}^N \mathsf{E}_\mathsf{Q} W_n = \mathsf{E}_\mathsf{Q} W.$$

An application of Proposition 4.2 yields the desired statement.

Step 4. Assume that each A_n is linear and let us prove the inclusion

$$\bigcap_{n=1}^N \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + Y_n) \cap \mathcal{R}_n \subseteq \mathcal{E}(X, Y).$$

Take Q from the left-hand side of this inclusion. Find $Y' \sim Y$ such that $\mathsf{E}_\mathsf{Q} Y'_n = 0$ for any n. Clearly, $\mathsf{Q} \in \bigcap_n \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + Y'_n) \cap \mathcal{R}_n$, so that

$$u_n(W_n + X_n + Y'_n) = \mathsf{E}_\mathsf{Q}(W_n + X_n + Y'_n) = \mathsf{E}_\mathsf{Q}(W_n + X_n), \quad n = 1, \dots, N.$$

As $\mathcal{X}_{\mathcal{D}_n}(W_n + X_n + Y'_n) \neq \emptyset$, we have (by the definition of $\mathcal{X}$) that $u_n(W_n + X_n + Y'_n) > -\infty$. Furthermore, (by the definition of $\mathcal{R}$) $\mathsf{E}_\mathsf{Q} X_n \le 0$, so that $X_n \in L^1(\mathsf{Q})$. For any n, $\xi \in A_n$, and $\eta \in L^1(\mathsf{Q})$ such that $\mathsf{E}_\mathsf{Q}\eta = 0$, we have

$$\begin{aligned} u_n(W_n + \xi + \eta) &\le \mathsf{E}_\mathsf{Q}(W_n + \xi + \eta) = \mathsf{E}_\mathsf{Q}(W_n + \xi) \\ &= \mathsf{E}_\mathsf{Q}(W_n + X_n + (\xi - X_n)) \le \mathsf{E}_\mathsf{Q}(W_n + X_n) = u_n(W_n + X_n + Y'_n) \end{aligned}$$

(in the second inequality, we used the linearity of A_n), so that (X, Y', Q) is an Arrow-Debreu-type equilibrium.

Step 5. Assume that each A_n is linear and let us prove the inclusion

$$\mathcal{E}(X,Y) \subseteq \bigcap_{n=1}^{N} \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + Y_n) \cap \mathcal{R}_n.$$

Take $\mathsf{Q} \in \mathcal{E}(X,Y)$ and find $Y' \sim Y$ such that (X, Y', Q) is an Arrow-Debreu-type equilibrium. It was shown in Step 2 that $\mathsf{Q} \in \bigcap_n \mathcal{D}_n \cap \mathcal{R}_n$. Applying (4.4) and Proposition 2.6, we get

$$\begin{aligned} \mathsf{E}_{\mathsf{Q}} W_n &= \sup_{\eta \in L^1(\mathsf{Q}) : \mathsf{E}_{\mathsf{Q}}\eta = 0} u_n(W_n + \eta) \le u_n(W_n + X_n + Y'_n) \\ &\le \mathsf{E}_{\mathsf{Q}}(W_n + X_n + Y'_n) \le \mathsf{E}_{\mathsf{Q}} W_n, \quad n = 1, \dots, N. \end{aligned}$$

Consequently, $u_n(W_n + X_n + Y'_n) = \mathsf{E}_{\mathsf{Q}}(W_n + X_n + Y'_n)$ for any n, which means that $\mathsf{Q} \in \bigcap_n \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + Y'_n) = \bigcap_n \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + Y_n)$. □

The assumption that each A_n is linear is essential for the second part of Theorem 4.7 as shown by the following example.

Example 4.8. Let $N = 2$, $\mathcal{D}_1 = \mathcal{D}_2 = \{\mathsf{P}\}$, $A_1 = A_2 = \mathbb{R}_-$ (i.e. A_1, A_2 consist of random variables that are identically equal to a negative constant), and $W_1 = W_2 = 0$. Take $X_1 = X_2 = -1$, $Y_1 = Y_2 = 0$. Then $\bigcap_n \mathcal{X}_{\mathcal{D}_n}(X_n + Y_n) \cap \mathcal{R}_n = \{\mathsf{P}\}$, but clearly (X, Y) is not a Pareto-type equilibrium. □

By Theorem 4.7, the set $\mathcal{E}(X,Y)$ does not depend on (X,Y). We call it the set of *equilibrium measures* and denote by $\mathcal{E}$ (Theorem 4.7 yields the representation $\mathcal{E} = \mathcal{X}_{\bigcap_n \mathcal{D}_n \cap \mathcal{R}_n}(W)$).

From the financial point of view, $\mathcal{E}$ is the set of equilibrium price systems. Thus, it is natural to define the set of *unconstrained equilibrium prices* of a contingent claim F simply as $I_E(F) := \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{E}\}$.

4.2 Constrained Equilibrium

Let u_n, $\mathcal{D}_n$, A_n, and W_n be the same as in the previous subsection. Let S be a d-dimensional random vector whose components belong to $\bigcap_n L^1_s(\mathcal{D}_n)$. From the financial point of view, there are d contracts that can be exchanged between the agents, and S^i means the discounted payoff of the i-th contract (for example, if the i-th contract is a share, S^i is its discounted price at time 1). There is no relation between S and A_n; A_n means the set of P&Ls that can be obtained by the n-th agent by trading on a "big market", without trading the assets $1, \dots, d$ (if there is no "big market", i.e. the whole market consists of these agents and these contracts, then $A_n = 0$).

Definition 4.9. The *maximal overall utility* is defined as

$$M = \sup_{\substack{X_n \in A_n \\ h_n \in \mathbb{R}^d : \sum_n h_n = 0}} \sum_{n=1}^{N} u_n(W_n + X_n + \langle h_n, S \rangle),$$

where the sum is understood as $-\infty$ if any of the summands equals $-\infty$.

Let us introduce the notation

$$\begin{aligned} G_n &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}} S : \mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n\}, \\ \widetilde{G}_n &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}}(S, W_n) : \mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n\}, \\ f_n(x) &= \inf\{y : (x, y) \in \widetilde{G}_n\}, \quad x \in G_n, \\ G &= \bigcap_{n=1}^{N} G_n, \\ f(x) &= \sum_{n=1}^{N} f_n(x), \quad x \in G. \end{aligned}$$

Proposition 4.10. *We have*

$$M = \inf_{x \in G} f(x),$$

where $\inf \emptyset := \infty$.

`Proof`. By Proposition 2.6, we have for any $h_n \in \mathbb{R}^d$,

$$\begin{aligned} \sup_{X_n \in A_n} u_n(W_n + X_n + \langle h_n, S \rangle) &= \inf_{\mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n} \mathsf{E}_{\mathsf{Q}}(W_n + \langle h_n, S \rangle) \\ &= \inf_{x \in \widetilde{G}_n} \langle (h_n, 1), x \rangle \\ &= \inf_{x \in G_n} (\langle h_n, x \rangle + f_n(x)). \end{aligned}$$

Standard results of convex analysis (see [24; Th. 16.4]) yield

$$M = \sup_{h_n \in \mathbb{R}^d : \sum_n h_n = 0} \sum_{n=1}^{N} \inf_{x \in G_n} (\langle h_n, x \rangle + f_n(x)) = \inf_{x \in G} f(x).$$

We now introduce two definitions of equilibrium.

Definition 4.11. A *Pareto-type equilibrium* is a collection $(X, h) = (X_1, \dots, X_N, h_1, \dots, h_N)$ such that

(a) $X_n \in A_n$;
(b) $h_n \in \mathbb{R}^d$, $\sum_n h_n = 0$;
(c) there do not exist (X', h') satisfying (a), (b) and such that

$$\begin{aligned} &\forall n,\ u_n(W_n + X_n' + \langle h_n', S \rangle) \geq u_n(W_n + X_n + \langle h_n, S \rangle), \\ &\exists n : u_n(W_n + X_n' + \langle h_n', S \rangle) > u_n(W_n + X_n + \langle h_n, S \rangle). \end{aligned}$$

`Remark`. It is easy to see from the translation invariance property ($u_n(X + m) = u_n(X) + m$) that condition (c) is equivalent to:

(c') $\sum_n u_n(W_n + X_n + \langle h_n, S \rangle) = M$.

Definition 4.12. An *Arrow-Debreu-type equilibrium* is a collection (X, h, P), where $P \in \mathbb{R}^d$, such that

(a) $X_n \in A_n$;

(b) $h_n \in \mathbb{R}^d$, $\sum_n h_n = 0$;
(c) for any n,

$$u_n(W_n + X_n + \langle h_n, S - P\rangle) = \max_{\xi \in A_n,\, \eta \in \mathbb{R}^d} u_n(W_n + \xi + \langle \eta, S - P\rangle).$$

Let us introduce the notation

$$E(X,h) = \{P \in \mathbb{R}^d : (X,h,P) \text{ is an Arrow-Debreu-type equilibrium}\}.$$

Theorem 4.13. *Assume that $M < \infty$ (by Proposition 4.10, this is equivalent to: $G \neq \emptyset$). Let $X_n \in A_n$, $h_n \in \mathbb{R}^d$, $\sum_n h_n = 0$. The following conditions are equivalent:*
(i) *(X,h) is a Pareto-type equilibrium;*
(ii) *there exists $P \in \mathbb{R}^d$ such that (X,h,P) is an Arrow-Debreu-type equilibrium.*
If these conditions are satisfied, then

$$E(X,h) = \operatorname*{argmin}_{x \in G} f(x).$$

If each A_n is a linear space, then (i), (ii) *are equivalent to:*
(iii) $\bigcap_n \{\mathsf{E_Q} S : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + \langle h_n, S\rangle) \cap \mathcal{R}_n\} \neq \emptyset$.
Moreover, in this case

$$E(X,h) = \bigcap_{n=1}^{N} \{\mathsf{E_Q} S : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + \langle h_n, S\rangle) \cap \mathcal{R}_n\}.$$

Proof. *Step 1.* Let us prove the implication (i)$\Rightarrow$(ii). Take $P \in \operatorname{argmin}_{x \in G} f(x)$. Using Proposition 4.10, we can write

$$\begin{aligned}
\sum_{n=1}^{N} u_n(W_n + X_n + \langle h_n, S\rangle) &\le \sum_{n=1}^{N} \inf_{\mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n} \mathsf{E_Q}(W_n + \langle h_n, S\rangle) \\
&= \sum_{n=1}^{N} \inf_{x \in \widetilde{G}_n} \langle (h_n, 1), x\rangle \\
&\le \sum_{n=1}^{N} (f_n(P) + \langle h_n, P\rangle) = f(P) = M.
\end{aligned}$$

As the left-hand side and the right-hand side of this inequality coincide, we get $u_n(W_n + X_n + \langle h_n, S\rangle) = f_n(P) + \langle h_n, P\rangle$ for any n. Thus, for any n, $\xi \in A_n$, and $\eta \in \mathbb{R}^d$, we have

$$\begin{aligned}
u_n(W_n + \xi + \langle \eta, S - P\rangle) &= u_n(W_n + \xi + \langle \eta, S\rangle) - \langle \eta, P\rangle \\
&\le \inf_{\mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n} \mathsf{E_Q}(W_n + \langle \eta, S\rangle) - \langle \eta, P\rangle \\
&= \inf_{x \in \widetilde{G}_n} \langle (\eta, 1), x\rangle - \langle \eta, P\rangle \\
&\le f_n(P) + \langle \eta, P\rangle - \langle \eta, P\rangle = f_n(P) \\
&= u_n(W_n + X_n + \langle h_n, S - P\rangle).
\end{aligned}$$

Thus, (X,h,P) is an Arrow-Debreu-type equilibrium.

Step 2. The implication (ii)⇒(i) follows from the inequality: for any $\widetilde{X}_n \in A_n$ and $\widetilde{h}_n \in \mathbb{R}^d$ with $\sum_n \widetilde{h}_n = 0$, we have

$$\begin{aligned}\sum_{n=1}^{N} u_n(W_n + \widetilde{X}_n + \langle \widetilde{h}_n, S\rangle) &= \sum_{n=1}^{N} u_n(W_n + \widetilde{X}_n + \langle \widetilde{h}_n, S-P\rangle)\\ &\le \sum_{n=1}^{N} u_n(W_n + X_n + \langle h_n, S-P\rangle)\\ &= \sum_{n=1}^{N} u_n(W_n + X_n + \langle h_n, S\rangle).\end{aligned}$$

Step 3. It was shown in Step 1 that

$$\operatorname*{argmin}_{x\in G} f(x) \subseteq E(X,h).$$

Let us prove the reverse inclusion. Take $P \in E(X,h)$. Fix n and suppose that $P \notin G_n$. Then there exists $\eta \in \mathbb{R}^d$ such that

$$\langle \eta, P\rangle < \inf_{x\in G_n} \langle \eta, x\rangle = \inf_{\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}_n} \mathsf{E}_{\mathsf{Q}}\langle \eta, S\rangle.$$

According to Proposition 2.6, there exists $\xi \in A_n$ such that $\langle \eta, P\rangle < u_n(\xi + \langle \eta, S\rangle)$. Then

$$u_n(W_n + \alpha\xi + \langle \alpha\eta, S-P\rangle) \xrightarrow[\alpha\to\infty]{} \infty. \tag{4.5}$$

In view of Proposition 4.10, the condition $M < \infty$ implies that $\mathcal{D}_n \cap \mathcal{R}_n \neq \emptyset$. Fix $\mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n$. Then

$$\begin{aligned}u_n(W_n + \alpha\xi + \langle \alpha\eta, S-P\rangle) &\le u_n(W_n + X_n + \langle h_n, S-P\rangle)\\ &\le \mathsf{E}_{\mathsf{Q}}(W_n + \langle h_n, S-P\rangle) < \infty,\end{aligned}$$

which contradicts (4.5). Thus, $P \in G$. Using Proposition 2.6, we can write

$$\begin{aligned}M &\ge \sum_{n=1}^{N} u_n(W_n + X_n + \langle h_n, S-P\rangle)\\ &= \sum_{n=1}^{N} \sup_{\xi\in A_n,\eta\in\mathbb{R}^d} u_n(W_n + \xi + \langle \eta, S-P\rangle)\\ &= \sum_{n=1}^{N} \sup_{\eta\in\mathbb{R}^d} \inf_{\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}_n} \mathsf{E}_{\mathsf{Q}}(W_n + \langle \eta, S-P\rangle)\\ &= \sum_{n=1}^{N} \sup_{\eta\in\mathbb{R}^d} \inf_{x\in G_n} (f_n(x) + \langle \eta, x-P\rangle)\\ &= \sum_{n=1}^{N} f_n(P) = f(P).\end{aligned}$$

An application of Proposition 4.10 yields the desired statement.

Step 4. Assume that each A_n is linear and let us prove the inclusion

$$\bigcap_{n=1}^N \{\mathsf{E}_\mathsf{Q} S : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + \langle h_n, S\rangle) \cap \mathcal{R}_n\} \subseteq E(X, h).$$

Take P from the left-hand side of this inclusion. Using the same arguments as in the proof of [6; Prop. 2.9], we can find for every n a measure $\mathsf{Q}_n \in \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + \langle h_n, S\rangle) \cap \mathcal{R}_n$ such that $P = \mathsf{E}_{\mathsf{Q}_n} S$ (using the $\mathcal{D}_n$-consistency of A_n, it is easy to check that $\mathcal{X}_{\mathcal{D}_n}(W_n + X_n + \langle h_n, S\rangle) \cap \mathcal{R}_n$ is L^1-closed, so this set is weakly compact). Then

$$\begin{aligned} u_n(W_n + X_n + \langle h_n, S - P\rangle) &= \mathsf{E}_{\mathsf{Q}_n}(W_n + X_n + \langle h_n, S\rangle) - \langle h, P\rangle \\ &= \mathsf{E}_{\mathsf{Q}_n}(W_n + X_n), \quad n = 1, \dots, N. \end{aligned}$$

As $\mathcal{X}_{\mathcal{D}_n}(W_n + X_n + \langle h_n, S\rangle) \neq \emptyset$, we have (by the definition of $\mathcal{X}$) that $u_n(W_n + X_n + \langle h_n, S\rangle) > -\infty$. Furthermore, (by the definition of $\mathcal{R}$) $\mathsf{E}_{\mathsf{Q}_n} X_n \le 0$, so that $X_n \in L^1(\mathsf{Q}_n)$. For any n, $\xi \in A_n$, and $\eta \in \mathbb{R}^d$, we have

$$\begin{aligned} u_n(W_n + \xi + \langle \eta, S - P\rangle) &\le \mathsf{E}_{\mathsf{Q}_n}(W_n + \xi + \langle \eta, S - P\rangle) = \mathsf{E}_{\mathsf{Q}_n}(W_n + \xi) \\ &= \mathsf{E}_{\mathsf{Q}_n}(W_n + X_n + (\xi - X_n)) \le \mathsf{E}_{\mathsf{Q}_n}(W_n + X_n) \\ &= u_n(W_n + X_n + \langle h_n, S - P\rangle) \end{aligned}$$

(in the second inequality, we used the linearity of A_n), so that (X, h, P) is an Arrow-Debreu-type equilibrium.

Step 5. Assume that each A_n is linear and let us prove the inclusion

$$E(X, h) \subseteq \bigcap_{n=1}^N \{\mathsf{E}_\mathsf{Q} S : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + \langle h_n, S\rangle) \cap \mathcal{R}_n\}.$$

Take $P \in E(X, h)$. It was shown in Step 3 that $P \in G$. Using the same arguments as in the proof of [6; Prop. 2.9], we can find for every n a measure $\mathsf{Q}_n \in \mathcal{D}_n \cap \mathcal{R}_n$ such that $\mathsf{E}_{\mathsf{Q}_n} S = P$ and $\mathsf{E}_{\mathsf{Q}_n} W_n = f_n(P)$. Applying Proposition 2.6, we get

$$\begin{aligned} f_n(P) &= \sup_{\eta \in \mathbb{R}^d} \inf_{x \in G_n} (f_n(x) + \langle \eta, x - P\rangle) \\ &= \sup_{\eta \in \mathbb{R}^d} \inf_{\mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n} \mathsf{E}_\mathsf{Q}(W_n + \langle \eta, S - P\rangle) \\ &= \sup_{\xi \in A_n, \eta \in \mathbb{R}^d} u_n(W_n + \xi + \langle \eta, S - P\rangle) \\ &= u_n(W_n + X_n + \langle h_n, S - P\rangle) \\ &\le \mathsf{E}_{\mathsf{Q}_n}(W_n + X_n + \langle h_n, S - P\rangle) \\ &\le \mathsf{E}_{\mathsf{Q}_n} W_n = f_n(P). \end{aligned}$$

Consequently, $u_n(W_n + X_n + \langle h_n, S - P\rangle) = \mathsf{E}_{\mathsf{Q}_n}(W_n + X_n + \langle h_n, S - P\rangle)$ for any n, which means that $\mathsf{Q}_n \in \mathcal{X}_{\mathcal{D}_n}(W_n + X_n + \langle h_n, S\rangle)$. □

Let G_n° denote the relative interior of G_n, and, for $P \in G_n$, we denote

$$N_{\widetilde{G}_n}(P) = \bigl\{\eta \in \mathbb{R}^d : \langle (\eta, 1), (P, f_n(P))\rangle = \inf_{x \in \widetilde{G}_n} \langle (\eta, 1), x\rangle\bigr\}.$$

If $\widetilde{G}_n$ has a non-empty interior, then $N_{\widetilde{G}_n}(P)$ is the set of vectors $\eta \in \mathbb{R}^d$ such that $(\eta, 1)$ is an inner normal to $\widetilde{G}_n$ at the point $(P, f_n(P))$.

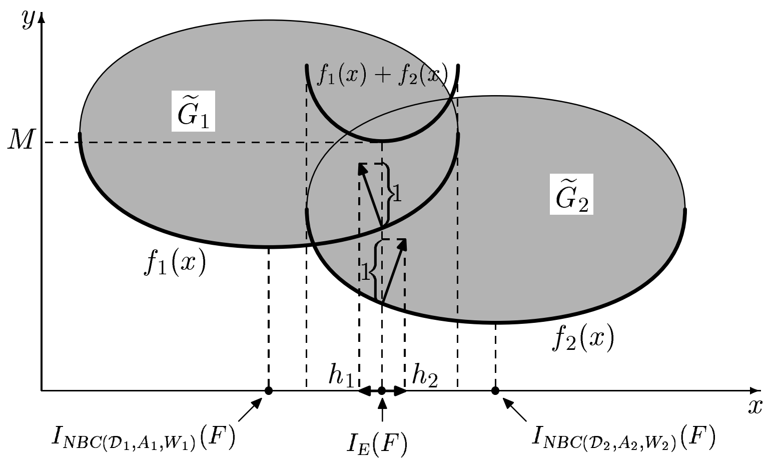

Figure 10. Geometric solution of the constrained equilibrium problem. Here $S=F$ is a one-dimensional contingent claim. The figure shows the maximal overall utility M, the equilibrium holdings h_n, and the equilibrium price $I_E(F)$. It also shows the NBC prices $I_{NBC(\mathcal{D}_n,A_n,W_n)}(F)$ of different agents.

Theorem 4.14. *Assume that* $\bigcap_n G_n^\circ \neq \emptyset$. *Take* $P\in \operatorname{argmin}_{x\in G} f(x)$. *Then there exist* $h_n \in N_{\widetilde{G}_n}(P)$ *such that* $\sum_n h_n = 0$. *Assume that, for each* n, *there exists* $X_n \in \operatorname{argmax}_{\xi\in A_n} u_n(W_n+\xi+\langle h_n, S\rangle)$. *Then* $(X_1,\dots,X_N,h_1,\dots,h_N,P)$ *is an Arrow-Debreu-type equilibrium. Conversely, any Arrow-Debreu-type equilibrium has such a form.*

`Proof`. Denote $f_n^*(\eta)=\inf_{x\in G_n}(f_n(x)+\langle \eta,x\rangle)$, $\eta\in\mathbb{R}^d$. Standard results of convex analysis (see [24; Th. 16.4]) guarantee that there exist $h_1,\dots,h_N\in\mathbb{R}^d$ such that $\sum_n h_n=0$ and $\sum_n f_n^*(h_n)=f(P)$. It follows from the line

$$f(P)=\sum_{n=1}^N f_n^*(h_n)\le \sum_{n=1}^N (f_n(P)+\langle h_n,P\rangle)=f(P)$$

that $h_n\in N_{\widetilde{G}_n}(P)$ for any n. By Proposition 2.6,

$$\begin{aligned}
u_n(W_n+X_n+\langle h_n,S\rangle) &= \sup_{\xi\in A_n} u_n(W_n+\xi+\langle h_n,S\rangle)\\
&= \inf_{\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}_n} \mathsf{E}_{\mathsf{Q}}(W_n+\langle h_n,S\rangle)\\
&= \inf_{x\in\widetilde{G}_n}\langle (h_n,1),x\rangle\\
&= f_n^*(h_n)=f_n(P)+\langle h_n,P\rangle,\quad n=1,\dots,N.
\end{aligned}$$

Consequently, for any n, $\xi\in A_n$, and $\eta\in\mathbb{R}^d$,

$$\begin{aligned}
u_n(W_n+\xi+\langle \eta,S-P\rangle) &\le \inf_{\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}_n} \mathsf{E}_{\mathsf{Q}}(W_n+\langle \eta,S-P\rangle)\\
&\le \inf_{x\in\widetilde{G}_n}\langle (\eta,1),x\rangle-\langle \eta,P\rangle\\
&= f_n^*(\eta)-\langle \eta,P\rangle\le f_n(P)\\
&= u_n(W_n+X_n+\langle h_n,S-P\rangle),
\end{aligned}$$

so that (X,h,P) is an Arrow-Debreu-type equilibrium.

Conversely, let (X,h,P) be an Arrow-Debreu-type equilibrium. According to Proposition 4.10, $P \in \operatorname{argmin}_{x\in G} f(x)$. Using the same arguments as in Step 1 of the proof of Theorem 4.13, we deduce that $h_n \in N_{\widetilde{G}_n}(P)$ for any n. The inclusion $X_n \in \operatorname{argmax}_{\xi\in A_n} u_n(W_n + \xi + \langle h_n, S\rangle)$ is clear from the definition of the Arrow-Debreu-type equilibrium. □

By Theorem 4.13, the set $E(X,Y)$ does not depend on (X,Y). We call it the set of *equilibrium prices* and denote by E (Theorem 4.13 yields the representation $E = \operatorname{argmin}_{x\in G} f(x)$).

From the financial point of view, E is the set of equilibrium price vectors for the multidimensional contract S. If $S = F$ is a one-dimensional contingent claim, we call E the set of *constrained equilibrium prices* of F and denote it by $I_E(F)$.

5 Comparison of Various Pricing Techniques

In [6] and in the present paper, we have proposed seven different pricing techniques based on coherent risks. They differ by the inputs they require and by the ideas behind them. These techniques are compared by Figure 11 and by Table 1.

The first, the second, and the fifth techniques typically provide the whole interval of fair prices, while the other techniques typically provide a unique price.

When applying the second and the third techniques, we can employ risk measures that are used in practice to measure *risk*, like Tail V@R of order 0.05. However, for the other five techniques one should use from the outset much more "moderate" risk measures. The reason is that all the pricing kernels should be very close to the original probability measure. One choice is to take $\mathcal{D}$ (or $\mathcal{D}_n$) as $\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}$, where $\mathcal{PD}$ is the set of possible historic measures (a typical choice is that it is a singleton) and $\mathcal{RD}$ is the determining set of a risk measure, like Tail V@R of order 0.05. Another choice is that $\mathcal{D}$ (or $\mathcal{D}_n$) is the determining set of a measure, like Tail V@R of order close to 1.

The first six techniques provide a kernel estimate of a price. Thus, they can also be applied to the estimation of sensitivity coefficients. The idea is as follows. Consider, as an example, an option with the discounted payoff $F = f(S_1)$, where S_1 is he terminal price of some asset and f is a smooth function. The true value of this option is

$$V = \mathsf{E}_{\mathsf{Q}} f(S_1) = \mathsf{E}_{\mathsf{Q}} f(S_0(1+r)),$$

where Q is the true valuation measure used by the market, $r = (S_1 - S_0)/S_0$, and S_0 is the initial price of the asset. The measure Q is not known exactly, but each of the first six techniques says that Q belongs to some set $\mathcal{Q}$ (for example, the utility-based NGD yields $\mathcal{Q} = \mathcal{D} \cap \mathcal{R}$). The true value of the option's delta is

$$\Delta = \frac{\partial V}{\partial S_0} = \mathsf{E}_{\mathsf{Q}}(1+r)f'(S_0(1+r)).$$

As we do not know Q exactly, but only know a set, to which it belongs, we get the following interval for deltas:

$$I = \{\mathsf{E}_{\mathsf{Q}}(1+r)f'(S_0(1+r)) : \mathsf{Q} \in \mathcal{Q}\}.$$

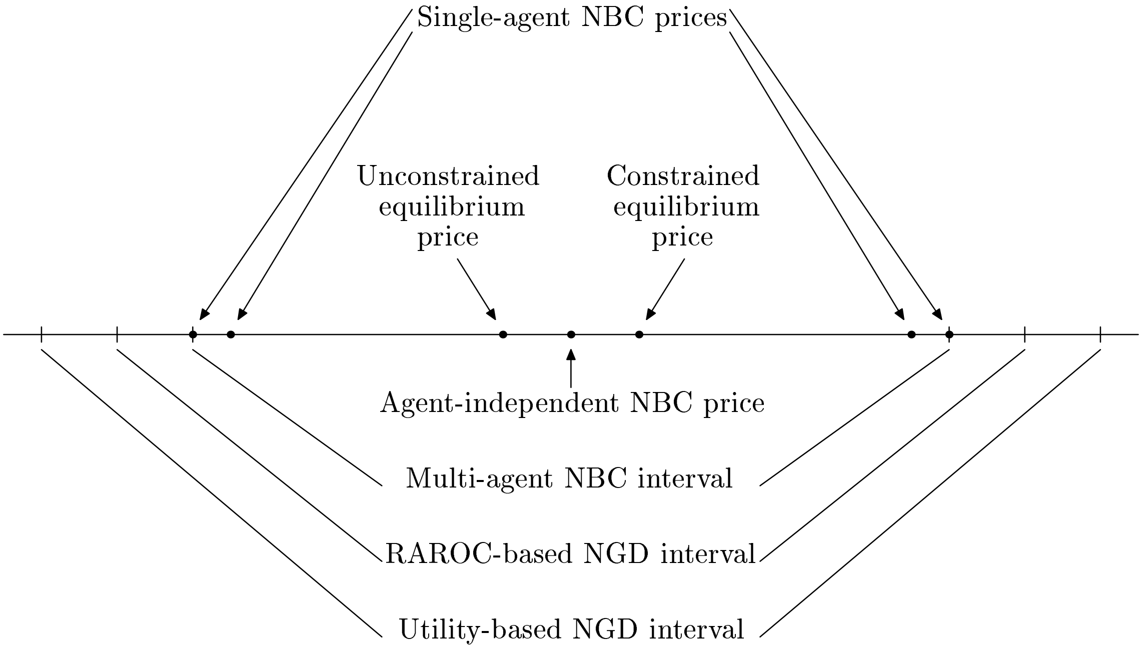

Figure 11. The form of fair prices provided by various techniques

Pricing technique	Inputs	Form of the price interval
Utility-based No Good Deals	$\mathcal{D}$, A	$\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}$
RAROC-based No Good Deals	$\mathcal{PD}$, $\mathcal{RD}$, A, R	$\left\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \left(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\right) \cap \mathcal{R}\right\}$
Agent-independent No Better Choice	$\mathcal{PD}$, $\mathcal{RD}$, A	$\left\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \left(\frac{1}{1+R_*}\mathcal{PD} + \frac{R_*}{1+R_*}\mathcal{RD}\right) \cap \mathcal{R}\right\}$, where $R_* = \sup_{X \in A} \mathrm{RAROC}(X)$
		$\left\{\frac{1}{1+R_*}\mathsf{E}_{\mathsf{P}} F + \frac{R_*}{1+R_*}\mathsf{E}_{\mathsf{Q}_*} F\right\}$, if $\mathcal{PD} = \{\mathsf{P}\}$ and $\mathcal{X}_{\mathcal{RD}}(X_*) = \{\mathsf{Q}_*\}$, where $X_* \in \mathrm{argmax}_{X \in A} \mathrm{RAROC}(X)$
Single-agent No Better Choice	$\mathcal{D}$, A, W	$\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}}(W) \cap \mathcal{R}\}$
		$\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}}(W)\}$ provided that this is a singleton and $u(W) = \max_{X \in A} u(W+X)$
Multi-agent No Better Choice	$\mathcal{D}_1, \dots, \mathcal{D}_N$, A, $W_1, \dots, W_N$	$\left\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathrm{conv}_{n=1}^{N}(\mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R})\right\}$
Unconstrained equilibrium	$\mathcal{D}_1, \dots, \mathcal{D}_N$, $A_1, \dots, A_N$, $W_1, \dots, W_N$	$\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{E}\}$, where $\mathcal{E} = \mathcal{X}_{\bigcap_n \mathcal{D}_n \cap \mathcal{R}_n}\left(\sum_n W_n\right)$
Constrained equilibrium	$\mathcal{D}_1, \dots, \mathcal{D}_N$, $A_1, \dots, A_N$, $W_1, \dots, W_N$	$\mathrm{argmin}_x \sum_n f_n(x)$, where $f_n(x) = \inf\{\mathsf{E}_{\mathsf{Q}} W_n : \mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}_n,\ \mathsf{E}_{\mathsf{Q}} F = x\}$

Table 1. The form of fair prices provided by various techniques

References

[1] *C. Acerbi.* Coherent representations of subjective risk aversion. In: G. Szegö (Ed.). Risk measures for the 21st century. Wiley, 2004, pp. 147–207.

[2] *Y. Ait-Sahalia, A. Lo.* Nonparametric risk management and implied risk aversion. Journal of Econometrics, **94** (2000), No. 1, p. 9–51.

[3] *P. Barrieu, N. El Karoui.* Inf-convolution of risk measures and optimal risk transfer. Finance and Stochastics, **9** (2005), p. 269–298.

[4] *F. Black.* Estimating expected return. Financial Analysts Journal, 1995, p. 168–171.

[5] *P. Carr, H. Geman, D. Madan.* Pricing and hedging in incomplete markets. Journal of Financial Economics, **62** (2001), p. 131–167.

[6] *A.S. Cherny.* Pricing with coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[7] *A.S. Cherny.* Weighted V@R and its properties. Finance and Stochastics, **10** (2006), No. 3, p. 367–393.

[8] *A.S. Cherny, D.B. Madan.* Coherent measurement of factor risks. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[9] *A.S. Cherny, D.B. Madan.* CAPM, rewards, and empirical asset pricing based on coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[10] *J. Cvitanić, I. Karatzas.* On dynamic measures of risk. Finance and Stochastics, **3** (1999), p. 451–482.

[11] *F. Delbaen.* Coherent monetary utility functions. Preprint, available at `http://www.math.ethz.ch/~delbaen` under the name "Pisa lecture notes".

[12] *D. Filipovic, M. Kupper.* Optimal Capital and Risk Transfers for Group Diversification. To be published in Mathematical Finance, available at: `http://www.math.ethz.ch/~kupper`.

[13] *D. Filipovic, M. Kupper.* Equilibrium Prices for Monetary Utility Functions. Preprint, available at: `http://www.math.ethz.ch/ kupper`.

[14] *T. Fischer.* Risk capital allocation by coherent risk measures based on one-sided moments. Insurance: Mathematics and Economics **32** (2003), No. 1, p. 135–146.

[15] *D. Heath, H. Ku.* Pareto equilibria with coherent measures of risk. Mathematical Finance, **14** (2004), p. 163–172.

[16] *J. Jackwerth.* Recovering risk aversion from option prices and realized returns. Review of Financial Studies, **13** (2000), No. 2, p. 433–451.

[17] *A. Jobert, A. Platania, L.C.G. Rogers.* A Bayesian solution to the equity premium puzzle. Preprint, available at: `http://www.statslab.cam.ac.uk/~chris`.

[18] *E. Jouini, W. Schachermayer, N. Touzi.* Optimal risk sharing for law invariant monetary utility functions. Preprint, available at: `http://www.fam.tuwien.ac.at/~wschach/pubs`.

[19] *T.C. Koopmans.* Three essays on the state of economic science. A.M. Kelley, 1990.

[20] *E.L. Lehmann.* Testing statistical hypotheses. Springer, 1997.

[21] *H. Markowitz.* Portfolio selection. Wiley, 1959.

[22] *Y. Nakano.* Minimizing coherent risk measures of shortfall in discrete-time models under cone constraints. Applied Mathematical Finance, **10** (2003), p. 163–181.

[23] *Y. Nakano.* Minimization of shortfall risk in a jump-diffusion model. Statistics and Probability Letters, **67** (2004), p. 87–95.

[24] *R.T. Rockafellar.* Convex analysis. Princeton, 1997.

[25] *R.T. Rockafellar, S. Uryasev.* Optimization of conditional Value-At-Risk. Journal of Risk, **2** (2000), No. 3, p. 21–41.

[26] *R.T. Rockafellar, S. Uryasev, M. Zabarankin.* Master funds in portfolio analysis with general deviation measures. Journal of Banking and Finance, **29** (2005).

[27] *J. Rosenberg, R. Engle.* Empirical pricing kernels. Journal of Financial Economics, **64** (2002), No. 3, p. 341–372.

[28] *M. Rubinstein, J. Jackwerth.* Recovering probabilities and risk aversion from option prices and realized returns. In: B. Lehmann (Ed.). The legacy of Fisher Black. Oxford, 2004.

[29] *J. Sekine.* Dynamic minimization of worst conditional expectation of shortfall. Mathematical Finance, **14** (2004), No. 4, p. 605–618.

[30] *W. Sharpe.* Capital asset prices: A theory of market equilibrium under conditions of risk. Journal of Finance, **19** (1964), 425-442.