PRICING AND HEDGING EUROPEAN OPTIONS WITH DISCRETE-TIME COHERENT RISK

Alexander S. Cherny

Moscow State University
Faculty of Mechanics and Mathematics
Department of Probability Theory
119992 Moscow, Russia
E-mail: alexander.cherny@gmail.com
Webpage: http://mech.math.msu.su/~cherny

Abstract. The aim of the paper is to provide as explicit as possible expressions for upper/lower prices and for superhedging/subhedging strategies based on discrete-time coherent risk measures. This is done on 3 levels of generality. For a general infinite-dimensional model, we prove the fundamental theorem of asset pricing. For a general multidimensional model, we provide expressions for prices and hedges. For a wide class of models, including, in particular, GARCH, we give more concrete formulas, a sufficient condition for the uniqueness of a hedging strategy as well as a numerical algorithm.

Key words and phrases. Dynamic coherent risk measure, dynamic Tail V@R, dynamic Weighted V@R, fundamental theorem of asset pricing, hedging cash flow streams, Markov model, No Good Deals, price contribution, pricing cash flow streams, risk management, risk measurement.

1 Introduction

Overview. The problem of finding adequate price bounds in incomplete markets is one of the basic goals of the modern financial mathematics. The classical superreplication/subreplication price bounds are typically unacceptably wide. However, in practice an agent, who has sold an option, never tries to superreplicate it almost surely (because this is typically impossible), but rather tries to superreplicate it in such a way that the resulting risk remains within the limits prescribed by his/her management. Thus, a natural way to redefine the upper price is to replace the condition that the terminal wealth should be a.s. positive by the condition that the terminal wealth has a negative risk, the risk being measured in a more moderate way than simply by the worst case scenario. The hedging problem then becomes the problem of minimizing risk.

According to recent developments in financial mathematics, the right way to measure risk is through coherent or more general convex risk measures. Applications of these measures to pricing have already been studied in a number of papers. Let us mention [1], [2], [3], [6], [7], [8], [11], [12], [13], [15], [16], [17], [18], [19], [20], [21], [22], [25], [26], [27], [28]. These papers differ by the model under consideration (static, discrete-time, or continuous-time) and by the risk measure applied (coherent or convex, static or dynamic). Also, different papers have different objectives: some deal with the fundamental theorem of asset pricing, some study the dynamic consistency of prices, while some other study pricing and hedging.

In this paper, we will consider pricing and hedging in discrete-time models. In principle, the analysis of such models can be based both on the static and the dynamic risks.

We choose dynamic ones because they allow us to use a powerful tool of the backward induction. Another choice to be made is between the coherent and the convex risk measures. Although the latter class is more general, the former one leads to more explicit results, and for this reason, we restrict attention to coherent risks.

Goal of the paper. The major goal of the paper is to obtain as explicit as possible forms of the risk-based upper/lower prices as well as superhedging/subhedging strategies in the discrete-time setting. For a risk measure, we take the definition from [10], which is, in turn, an extension to unbounded processes of the definitions proposed by Cheridito et. al. [4], Cheridito and Kupper [5], and Jobert and Rogers [18].

First of all, we prove the fundamental theorem of asset pricing (FTAP) for the risk-based *No Good Deals* condition within the framework of a general infinite-dimensional model (Theorem 2.3).

Then we define the risk-based *upper* and *lower price processes* for a stream of cash flows within the framework of a general multidimensional model.[1] For price processes, we provide a probabilistic representation and a geometric one; for (risk-based) hedging strategies, we provide a geometric representation (Theorem 3.2).

From the practical point of view, it is important to have a fast method of estimating how the price of a large portfolio is altered after an additional contract has been added to it, without repeating the whole price calculation procedure for the new portfolio. This might be termed the *price contribution.* We provide an expression for this object (Corollary 3.4) based on results of [10].

As our major goal is obtaining as explicit as possible expressions for the price and the hedge, we then narrow the consideration to a model, in which (S, Θ) has the Markov property, where S is the price process and Θ is an auxiliary process (e.g., volatility). For example, models with (exponentially) independent increments, stochastic volatility models as well as the GARCH have this structure.[2] Within the framework of the Markov model, we get more concrete expressions for the price and the hedge as well as a simple sufficient condition for the uniqueness of a hedge (Theorem 4.6). The results are expressed through so-called A- and B-*operators* introduced and studied in the paper.

For a particularization of this model, in which the price process has exponentially independent increments and each cash flow is a convex/concave function of the underlying asset, we are able to provide an explicit form of the price, the hedge, and the price contribution (Theorem 4.8).

The results described above are theoretical. For the practical calculations, one needs a numerical procedure. For a discretized version of the Markov model, we provide such a procedure for calculating the price, the hedge, and the price contribution. It is a combination of the dynamic results of this paper with the static optimization method proposed by Pflug [23], Rockafellar and Uryasev [24]. The use of this method enables one to simplify the formulas and to increase the speed of computations.

Among the papers mentioned above, the ones dealing with the FTAP for coherent risks are Carr et. al. [2], Cherny [7], Jaschke and Küchler [17], and Staum [27], but all these papers deal with the static risk measures. The papers that study both the price

[1] We work with streams of cash flows because this enables us to consider not only single options, but also books of options; the general class of risk measures we use takes into account the timing of payments, and therefore, pricing a stream of cash flows is not equivalent to pricing its cumulative cash flow.

[2] Let us remark that both the technique of discrete-time pricing with coherent risk and the technique of continuous-time pricing with coherent risk have its advantages and disadvantages as compared to each other. One of the advantages of the former technique is that it enables one to employ the GARCH model, which has no proper continuous-time analog.

and the hedge and are applicable to discrete-time models are Föllmer and Leukert [16], Nakano [22], Roorda et al. [25], Sekine [26], and Xu [28]. However, [16], [22], [26], and [28] deal with the static risk measures, while [25] deals with a finite Ω of a special form and also differs from our paper at the level of definitions. In fact, our definitions of the price and the hedge are very close to the ones in Leitner [21], although there is an essential difference at the level of modeling: [21] deals with a continuous-time diffusion model and the BSDEs technique, while our paper deals with discrete time and the backward induction technique.

Our results admit a direct extension to pricing and hedging American options. This is the topic of the forthcoming paper.

Application. Our results are related to pricing and hedging, but also have an important interpretation from the viewpoint of risk measurement and management. When using a dynamic risk measure, a big problem is: what terminal date should be used as the basis of risk measurement? This problem seems to have no satisfactory solution, and a very unpleasant feature is that the choice of the horizon essentially affects risk (the dependence of risk on the horizon is linear or exponential; see [10; Ex. 3.4]). But on the other hand, if we have a portfolio of perfectly liquid assets, then its risk should not depend on the horizon since the portfolio can be liquidated at any time. Thus, a more correct way to analyze risk is through the *market-adjusted risk measure*, which is closely connected with the pricing and hedging problem. We show that if a portfolio consists only of perfectly liquid assets and perfectly illiquid ones, then risk measured this way gets the form:

$$\begin{array}{c}\text{Total risk}\\\text{of a portfolio}\end{array} = \begin{array}{c}\text{Static risk}\\\text{of liquid part}\end{array} + \begin{array}{c}\text{Dynamic risk}\\\text{of illiquid part.}\end{array} \tag{1.1}$$

In this application, pricing and hedging get the following interpretation:

$$\begin{aligned}\text{Pricing} &= \text{Risk measurement},\\ \text{Hedging} &= \text{Risk management},\\ \text{Price contribution} &= \text{Risk contribution}.\end{aligned}$$

Let us remark that one more way to look at the pricing and hedging problem is to interpret it as the coherent utility maximization problem. In this interpretation,

$$\begin{aligned}\text{Pricing} &= \text{Finding the optimal value},\\ \text{Hedging} &= \text{Finding the optimal strategy}.\end{aligned}$$

Structure of the paper. In Section 2, we prove the FTAP.

Section 3 deals with pricing and hedging in the general multidimensional model. Subsection 3.1 presents the general result. Subsection 3.2 is related to the price contribution. Subsection 3.3 describes the application to risk measurement and management.

Section 4 deals with pricing and hedging in the Markov model. The results are given in Subsection 4.2. They are expressed in terms of A- and B-operators, which are introduced and studied in Subsection 4.1. In Subsection 4.3, we consider a particularization of this model. Subsection 4.4 presents a numerical algorithm.

Section 5 concludes. Some technical results are gathered in the Appendix.

2 Fundamental Theorem of Asset Pricing

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P})$ be a filtered probability space. Let $\mathcal{D} = (\mathcal{D}_n)_{n=1,\dots,N}$ be a system of sets of random variables with the properties:

- any random variable Z from $\mathcal{D}_n$ is positive, $\mathcal{F}_n$-measurable, and satisfies the inequality $\mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) \le 1$;
- $\mathcal{D}_n$ is non-empty, L^1-closed, uniformly integrable, and $\mathcal{F}_{n-1}$-*convex*, i.e. for any $Z_1, Z_2 \in \mathcal{D}_n$ and any $[0,1]$-valued $\mathcal{F}_{n-1}$-measurable random variable λ, we have $\lambda Z_1 + (1-\lambda) Z_2 \in \mathcal{D}_n$.

Introduce the notation

$$\overline{\mathcal{D}} = \Big\{ \prod_{n=1}^{N} Z_n : Z_n \in \mathcal{D}_n \cup \{1\} \Big\}, \qquad \widetilde{\mathcal{D}} = \Big\{ \prod_{n=1}^{N} Z_n : Z_n \in \mathcal{D}_n \Big\}.$$

For a set $\mathcal{E}$ of random variables, we define the space

$$L^1(\mathcal{E}) = \big\{ X \in L^0 : \lim_{a\to\infty} \sup_{Z\in\mathcal{E}} \mathsf{E} Z|X| I(|X|>a) = 0 \big\},$$

where L^0 denotes the space of all random variables. It is easy to check that $L^1(\mathcal{E})$ is a linear space.

Let I be an arbitrary set and $(S_n^i)_{n=0,\dots,N,\, i\in I}$ be a family of $(\mathcal{F}_n)$-adapted processes such that $S_n^i \in L^1(\overline{\mathcal{D}})$ for any n, i. From the financial point of view, I is the set of traded assets and S_n^i is the discounted price of the i-th asset at time n.

The following definition was introduced in [10].

Definition 2.1. Let $X = (X_n)_{n=0,\dots,N}$ be a one-dimensional $(\mathcal{F}_n)$-adapted process. The *coherent utility* of X is the $[-\infty, \infty]$-valued process $u(X) = (u_n(X))_{n=0,\dots,N}$ defined as: $u_N(X) = 0$,

$$u_{n-1}(X) = \operatorname*{essinf}_{Z\in\mathcal{D}_n} \mathsf{E}(Z(X_n + u_n(X))\,|\,\mathcal{F}_{n-1}), \quad n = N, \dots, 1,$$

where $\mathsf{E}(\xi\,|\,\mathcal{G})$ is understood as $\mathsf{E}(\xi^+\,|\,\mathcal{G}) - \mathsf{E}(\xi^-\,|\,\mathcal{G})$ with the convention $\infty - \infty = -\infty$. The corresponding *coherent risk* is $\rho(X) = -u(X)$. The system $\mathcal{D}$ is called the *determining system* of u (or ρ).

The financial interpretation of the above definition is as follows: X describes a stream of cash flows, i.e. X_n is the discounted cash flow at time n; $\rho_n(X)$ is the risk at time n of the remaining part of the stream, i.e. $X_{n+1}, \dots, X_N$.

By $\mathcal{H}$ we will denote the set of predictable $\mathbb{R}^I$-valued processes H such that only a finite number of H^i differs from zero. From the financial point of view, $\mathcal{H}$ is the set of various trading strategies. The stream of cash flows corresponding to a strategy $H \in \mathcal{H}$ is

$$\langle H, \Delta S\rangle_n = \sum_{i\in I} H_n^i \Delta S_n^i, \quad n = 1, \dots, N,$$

where $\Delta S_n^i = S_n^i - S_{n-1}^i$.

Definition 2.2. **(i)** The model satisfies the *No Good Deals* (NGD) condition if $u_n(\langle H, \Delta S\rangle) \le 0$ for any $H \in \mathcal{H}$ and any n.

(ii) The model satisfies the *weak NGD* condition if $u_0(\langle H, \Delta S\rangle) \le 0$ for any $H \in \mathcal{H}$.

Introduce the notation

$$\mathcal{M}_n = \{Z \in L^0(\mathcal{F}_n) : \mathsf{E}(Z\Delta S_n^i \,|\, \mathcal{F}_{n-1}) = 0 \ \forall i \in I\}, \quad n = 1, \dots, N,$$
$$\mathcal{M} = \{\mathsf{Q} \ll \mathsf{P} : S^i \text{ is an } (\mathcal{F}_n, \mathsf{Q})\text{-martingale for any } i \in I\},$$

where $L^0(\mathcal{F}_n)$ is the space of $\mathcal{F}_n$-measurable random variables. We identify measures that are absolutely continuous with respect to P with their densities with respect to P.

We will say that $\mathcal{D}$ is *probabilistic* if $\mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) = 1$ for any n and any $Z \in \mathcal{D}_n$. We will say that $\mathcal{D}$ is *strictly positive* if $Z > 0$ a.s. for any n and any $Z \in \mathcal{D}_n$. As an example, the dynamic Weighted V@R with a weighting measure μ (see [10; Ex. 2.4] satisfies the first condition if and only if $\mu((0,1]) = 1$; it satisfies the second condition if and only if $\mu((1-\varepsilon, 1]) > 0$ for any $\varepsilon > 0$.

Theorem 2.3 (FTAP). (i) *The NGD is satisfied if and only if $\mathcal{D}_n \cap \mathcal{M}_n \neq \emptyset$ for any n.*

(ii) *If $\mathcal{D}$ is probabilistic, then the weak NGD is satisfied if and only if $\widetilde{\mathcal{D}} \cap \mathcal{M} \neq \emptyset$.*

(iii) *If $\mathcal{D}$ is strictly positive, then the NGD and the weak NGD are equivalent.*

`Proof.` **(i)** In order to prove the "only if" statement, fix n. Applying Lemma 2.4 to $\mathcal{G} = \mathcal{F}_{n-1}$, $\mathcal{E} = \mathcal{D}_n$, and

$$\mathcal{X} = \Big\{\sum_{i\in I} H^i \Delta S_n^i : H^i \in L^0(\mathcal{F}_{n-1}, [-1,1]) \text{ and } H^i \text{ differs from zero only for a finite number of } i\Big\}$$

(here $L^0(\mathcal{F}_{n-1}, [-1,1])$ denotes the set of $\mathcal{F}_{n-1}$-measurable $[-1,1]$-valued random variables), we get $\mathcal{D}_n \cap \mathcal{M}_n \neq \emptyset$.

In order to prove the "if" statement, fix $H \in \mathcal{H}$. Going backwards from N to 0, we easily check that $u_n(\langle H, \Delta S\rangle) \le 0$ for any n.

(ii) Let us prove the "only if" statement. We will prove by the induction in N a stronger one: the weak NGD implies the existence of $\mathsf{Q} \in \widetilde{\mathcal{D}} \cap \mathcal{M}$ such that $\mathsf{Q}|\mathcal{F}_0 = \mathsf{P}|\mathcal{F}_0$. Suppose the statement is true for $N-1$ and let us prove it for N. Denote by $\mathcal{A}$ the set of elements $A \in \mathcal{F}_1$, for which there exists $H \in \mathcal{H}$ such that $u_1(\langle H, \Delta S\rangle) > 0$ a.e. on A. Denote $\alpha = \sup\{\mathsf{P}(A) : A \in \mathcal{A}\}$ and find a sequence $A(k) \in \mathcal{A}$ with $\mathsf{P}(A(k)) \to \alpha$. Find $H(k)$ such that $u_1(\langle H(k), \Delta S\rangle) > 0$ a.e. on $A(k)$ and define

$$H = \sum_{k=1}^{\infty} H(k) I_{A(k)\setminus A(k-1)}.$$

Going backwards from N to 1, we check that

$$u_n(\langle H, \Delta S\rangle) = \sum_{k=1}^{\infty} u_n(\langle H(k), \Delta S\rangle) I_{A(k)\setminus A(k-1)}, \quad n = 1, \dots, N.$$

Consequently, $u_1(\langle H, \Delta S\rangle) > 0$ a.e. on $A = \bigcup_k A(k)$. Similarly, by the backward induction we prove that, for

$$\widetilde{H} = \frac{H}{u_1(\langle H, \Delta S\rangle)} I(u_1(\langle H, \Delta S\rangle) > 0),$$

we have $u_1(\langle \widetilde{H}, \Delta S\rangle) = I_A$.

Applying Lemma 2.4 to $\mathcal{G} = \mathcal{F}_0$, $\mathcal{E} = \mathcal{D}_1$, and

$$\mathcal{X} = \Big\{ \sum_{i\in I} H^i \Delta S_1^i + hI_A : H^i \in L^0(\mathcal{F}_0, [-1,1]),\ h \in L^0(\mathcal{F}_0, [0,1]), \text{ and } H^i \text{ differs from zero only for a finite number of } i \Big\},$$

we get $Z_1^* \in \mathcal{D}_1$ such that $\mathsf{E}(Z_1^* \Delta S_1^i | \mathcal{F}_0) = 0$ for any i and $Z_1^* = 0$ a.e. on A.

For any $H \in \mathcal{H}$, $u_1(\langle H, \Delta S\rangle) \le 0$ a.e. on A^c since otherwise, for $\overline{H} = \widetilde{H} I_A + H I_{A^c}$, we would have $\mathsf{P}(u_1(\langle \overline{H}, \Delta S\rangle) > 0) > \alpha$. Note that $\mathsf{P}(A^c) > 0$ due to the equality $Z_1^* I_A = 0$. Consider the model defined as $\Omega' = \Omega$, $\mathcal{F}' = \mathcal{F}$, $\mathcal{F}'_n = \mathcal{F}_{n-1}$, $\mathsf{P}' = \mathsf{P}(\,\cdot\,|A^c)$, $\mathcal{D}'_n = \mathcal{D}_{n-1}$, and $(S')^i_n = S^i_{n-1}$, $n = 0, \dots, N-1$. For this model, we have $u'_0(\langle H', \Delta S'\rangle) \le 0$ for any $H' \in \mathcal{H}'$. Applying the induction assumption, we get a measure $\mathsf{Q}' \ll \mathsf{P}'$ such that $\mathsf{Q}'|\mathcal{F}_1 = \mathsf{P}'|\mathcal{F}_1$ and the process $(S_n^i)_{n=1,\dots,N}$ is an $(\mathcal{F}_n, \mathsf{Q}')$-martingale for any i. Taking now $\mathsf{Q} = Z_1^* \mathsf{Q}'$ yields the desired result.

The "if" statement is proved in the same way as in (i).

(iii) Suppose that the NGD is violated, i.e. there exist $H \in \mathcal{H}$ and n_0 such that $\mathsf{P}(u_{n_0}(\langle H, \Delta S\rangle) > 0) > 0$. Consider

$$H'_n = \begin{cases} \dfrac{H_n}{u_{n_0}(\langle H, \Delta S\rangle)} I(u_{n_0}(\langle H, \Delta S\rangle) > 0), & n > n_0, \\ 0, & n \le n_0. \end{cases}$$

Then $u_{n_0}(\langle H', \Delta S\rangle) = I(u_{n_0}(\langle H, \Delta S\rangle) > 0)$. Using [10; Lem. 2.3] and going backwards from n_0 to 0, one can check that there exist $Z_1 \in \mathcal{D}_1, \dots, Z_{n_0} \in \mathcal{D}_{n_0}$ such that

$$u_n(\langle H', \Delta S\rangle) = \mathsf{E}(Z_{n+1} \dots Z_{n_0} u_{n_0}(\langle H', \Delta S\rangle) | \mathcal{F}_n), \quad n = 0, \dots, n_0.$$

From this representation, we see that $\mathsf{P}(u_0(\langle H', \Delta S\rangle) > 0) > 0$, i.e. the weak NGD is violated. □

Lemma 2.4. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, $\mathcal{E}$ be a non-empty $\mathcal{G}$-convex L^1-closed uniformly integrable set of random variables, and $\mathcal{X}$ be a convex subset of $L^1(\mathcal{E})$ such that $\operatorname{essinf}_{Z\in\mathcal{E}} \mathsf{E}(ZX|\mathcal{G}) \le 0$ for any $X \in \mathcal{X}$. Then there exists $Z_* \in \mathcal{E}$ such that $\mathsf{E}(Z_* X|\mathcal{G}) \le 0$ for any $X \in \mathcal{X}$.*

`Proof`. Fix a finite subset $\{X_1, \dots, X_K\}$ of $\mathcal{X}$ and a finite partition $\Omega = \bigsqcup_{m=1}^M A_m$ with $A_m \in \mathcal{G}$. Fix m and consider the set

$$C = \{\mathsf{E}(Z(X_1, \dots, X_K)|A_m) : Z \in \mathcal{E}\}.$$

By the Dunford-Pettis criterion, $\mathcal{E}$ is relatively weakly compact. As $\mathcal{E}$ is convex and L^1-closed, it is weakly closed by the Hahn-Banach theorem. Thus, $\mathcal{E}$ is weakly compact. As each map $\mathcal{E} \ni Z \mapsto \mathsf{E}(ZX_k|A_m)$ is weakly continuous, C is a convex compact in $\mathbb{R}^K$. Suppose that $C \cap (-\infty, 0]^K = \emptyset$. By the Hahn-Banach theorem, we can find $h \in \mathbb{R}^K$ such that

$$\sup_{x\in(-\infty,0]^K} \langle h, x\rangle \le 0 < \inf_{x\in C} \langle h, x\rangle.$$

Hence, $h \in \mathbb{R}_+^K$, and, by a slight move, we can choose $h \in (0,\infty)^K$. Without loss of generality, $\sum_i h^i = 1$. Then $X_0 := \sum_k h^k X_k \in \mathcal{X}$ and $\inf_{Z\in\mathcal{E}} \mathsf{E}(ZX_0|A_m) > 0$. According to [10; Lem. 2.3], there exists $Z_0 \in \mathcal{E}$ such that

$$\mathsf{E}(Z_0 X_0|\mathcal{G}) = \operatorname*{essinf}_{Z\in\mathcal{E}} \mathsf{E}(ZX_0|\mathcal{G}).$$

According to the conditions of the lemma, $\mathsf{E}(Z_0 X_0 | \mathcal{G}) \le 0$. But on the other hand, we should have $\mathsf{E}(Z_0 X_0 | A_m) > 0$. The obtained contradiction shows that, for any m, there exists $Z_m \in \mathcal{E}$ such that $\mathsf{E}(Z_m X_k | A_m) \le 0$ for any k. Then $Z := \sum_m Z_m I_{A_m} \in \mathcal{E}$ and $\mathsf{E}(Z X_k | \sigma(A_1, \dots, A_m)) \le 0$ for any k.

Thus, we have proved that, for any finite subset $\mathcal{Y}$ of $\mathcal{X}$ and any finite partition $\mathcal{A}$ of Ω, whose elements belong to $\mathcal{G}$, the set

$$B(\mathcal{A}, \mathcal{Y}) = \{Z \in \mathcal{E} : \mathsf{E}(ZX | \mathcal{A}) \le 0 \ \forall X \in \mathcal{Y}\}$$

is non-empty. Obviously, $B(\mathcal{A}, \mathcal{Y})$ is L^1-closed. Being convex, it is weakly closed by the Hahn-Banach theorem. Furthermore, any finite intersection of sets of this form contains a set of this form and thus is non-empty. As $\mathcal{E}$ is weakly compact, there exists Z_* that belongs to any set of this form. Clearly, Z_* is a desired element. □

Remark. Without the strict positivity of $\mathcal{D}$, the "if" statement in Theorem 2.3 (iii) remains valid, but the "only if" statement is no longer true. As an example, let $N = 2$, $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F}_0 = \text{triv}$, $\mathcal{F}_1 = \mathcal{F}_2 = 2^\Omega$, $\mathsf{P}(\omega_1) = \mathsf{P}(\omega_2) = 1/2$, $\mathcal{D}_1 = \{Z \in L^0(\mathcal{F}_1) : 0 \le Z \le 2, \mathsf{E}Z = 1\}$, $\mathcal{D}_2 = \{1\}$, $S_0 = S_1 = 0$, and $S_2 = I_{\{\omega_2\}}$. Consider the strategy $H_1 = 0$, $H_2 = I_{\{\omega_2\}}$. Then $u_1(\langle H, \Delta S\rangle) = I_{\{\omega_2\}}$, so that the NGD is violated. On the other hand, $2I_{\{\omega_1\}} \in \widetilde{\mathcal{D}} \cap \mathcal{M}$, so that the weak NGD is satisfied. □

3 Pricing and Hedging in the General Model

3.1 Pricing and Hedging

Consider the model of the previous section with a d-dimensional process S and let $F = (F_n)_{n=0,\dots,N}$ be a one-dimensional adapted process such that $F_n \in L^1(\overline{\mathcal{D}})$ for any n. The financial interpretation is as follows. We have assets of two types: totally liquid ones and totally illiquid ones. The discounted prices of liquid assets are given by S^i, $i = 1, \dots, d$, while the stream of cash flows produced by all the illiquid assets in a portfolio is given by F (so that F_n is the discounted amount received at time n). We will assume the NGD condition.

Definition 3.1. The *upper* and *lower price processes* of F are defined as

$$\overline{V}_n(F) = \operatorname{essinf}\{x \in L^0(\mathcal{F}_n) : \exists H \in \mathcal{H} : u_n(\langle H, \Delta S\rangle - F) + x \ge 0\}, \quad n = 0, \dots, N,$$
$$\underline{V}_n(F) = \operatorname{esssup}\{x \in L^0(\mathcal{F}_n) : \exists H \in \mathcal{H} : u_n(\langle H, \Delta S\rangle + F) - x \ge 0\}, \quad n = 0, \dots, N.$$

From the financial point of view, $\overline{V}_n(F)$ and $\underline{V}_n(F)$ are the discounted upper and lower prices at time n of the remaining part of the stream F, i.e. $F_{n+1}, \dots, F_N$. As there is complete symmetry between upper and lower prices, we can study only the lower one. This choice is made for the following reason: we consider F as a long position of an agent (this is convenient for the application to risk measurement). Thus, short positions are included in F with the minus sign. Then the lower price of F corresponds to the lower prices of long positions and the upper prices of short positions.

Let $\mathcal{C}$ be the set of non-empty convex compacts in $\mathbb{R}^{d+1}$. Consider the maps

$$\alpha(C) = \inf\{y \in \mathbb{R} : (0, y) \in C\}, \quad C \in \mathcal{C}, \tag{3.1}$$

$$\beta(C) = \big\{h \in \mathbb{R}^d : \langle (h,1), (0, \alpha(C))\rangle = \inf_{x \in C} \langle (h,1), x\rangle\big\}, \quad C \in \mathcal{C}. \tag{3.2}$$

The value $\alpha(C)$ is finite provided that $\mathrm{pr}_{\mathbb{R}^d} C$ contains 0, where $\mathrm{pr}_{\mathbb{R}^d}$ denotes the projection on the space generated by the first d coordinates. The set $\beta(C)$ is non-empty provided that $\mathrm{pr}_{\mathbb{R}^d} C$ contains 0 as an inner point.

Define the real-valued process $(\underline{a}_n(F))_{n=0,\dots,N}$, the $\mathcal{C}$-valued process $(\underline{G}_n(F))_{n=0,\dots,N}$, and the sequence of random sets $(\underline{\mathcal{H}}_n(F))_{n=1,\dots,N}$ by: $\underline{a}_N(F)=0$; if $\underline{a}_n(F),\dots,\underline{a}_N(F)$, $\underline{G}_n(F),\dots,\underline{G}_N(F)$, and $\underline{\mathcal{H}}_{n+1}(F),\dots,\underline{\mathcal{H}}_N(F)$ are already constructed, we set

$$\underline{G}_{n-1}(F) = \operatorname*{essconv}_{Z\in\mathcal{D}_n} \mathsf{E}(Z(\Delta S_n, F_n + \underline{a}_n(F))\,|\,\mathcal{F}_{n-1}), \tag{3.3}$$

$$\underline{a}_{n-1}(F) = \alpha(\underline{G}_{n-1}(F)), \tag{3.4}$$

$$\underline{\mathcal{H}}_n(F) = L^0(\mathcal{F}_{n-1}, \beta(\underline{G}_{n-1}(F))), \tag{3.5}$$

where essconv is the essential closed convex hull (see Appendix) and $L^0(\mathcal{F}_{n-1}, \beta(\underline{G}_{n-1}(F)))$ is the set of $\mathcal{F}_{n-1}$-measurable random vectors H such that $H \in \beta(\underline{G}_{n-1}(F))$ a.s. Going backwards from N to 0, we check that $\sup_{x\in\underline{G}_n(F)} \|x\| \in L^1(\overline{\mathcal{D}})$ and $\underline{a}_n(F) \in L^1(\overline{\mathcal{D}})$ for any n (the argument is exactly the same as in the proof of [10; Prop. 2.2]). In doing this, we employ Theorem 2.3, which tells us that $0 \in \mathrm{pr}_{\mathbb{R}^d} \underline{G}_n(F)$ a.s. for any n. If moreover 0 belongs to the interior of $\mathrm{pr}_{\mathbb{R}^d} \underline{G}_n(F)$ a.s. (as an example, the model of Subsection 4.2 satisfies this condition), then, according to Lemma A.2, $\underline{\mathcal{H}}_{n+1}(F) \neq \emptyset$.

By $\mathcal{D}\cap\mathcal{M}$ we will denote the system $(\mathcal{D}_n\cap\mathcal{M}_n)_{n=1,\dots,N}$, where $\mathcal{M}_n$ is defined in the previous section. It is easy to check that each $\mathcal{D}_n\cap\mathcal{M}_n$ is $\mathcal{F}_{n-1}$-convex, L^1-closed, and uniformly integrable. As the NGD is satisfied, each $\mathcal{D}_n\cap\mathcal{M}_n$ is non-empty due to Theorem 2.3.

By $u(X;\mathcal{D}')$ we denote the coherent utility of a process X corresponding to a determining system $\mathcal{D}'$.

Theorem 3.2. **(i)** *We have*

$$\underline{V}_n(F) = \underline{a}_n(F) = u_n(F;\mathcal{D}\cap\mathcal{M}), \quad n=0,\dots,N.$$

(ii) *If $H_n \in \underline{\mathcal{H}}_n(F)$ for any n, then*

$$u_n(\langle H, \Delta S\rangle + F) - \underline{V}_n(F) = 0, \quad n=0,\dots,N. \tag{3.6}$$

Conversely, if H is a predictable process, for which (3.6) *holds, then $H_n \in \underline{\mathcal{H}}_n(F)$ for any n. Moreover, if $\mathcal{D}$ is strictly positive, then it is sufficient here to require* (3.6) *only for $n=0$.*

`Proof.` **(i)** Going backwards from N to 0, we check that, for any $H\in\mathcal{H}$,

$$u_n(\langle H, \Delta S\rangle + F) \le u_n(\langle H, \Delta S\rangle + F;\mathcal{D}\cap\mathcal{M}) = u_n(F;\mathcal{D}\cap\mathcal{M}), \quad n=0,\dots,N.$$

Consequently, $\underline{V}_n(F) \le u_n(F;\mathcal{D}\cap\mathcal{M})$ for any n.

Fix $\varepsilon > 0$. Applying Lemmas A.1 and A.4, we get for any n a random vector $H_n \in L^0(\mathcal{F}_{n-1}, \mathbb{R}^d)$ such that

$$\operatorname*{essinf}_{Z\in\mathcal{D}_n} \mathsf{E}(Z(\langle H_n, \Delta S_n\rangle + F_n + \underline{a}_n(F))\,|\,\mathcal{F}_{n-1}) = \varphi_{(H_n,1)}(\underline{G}_{n-1}(F)) \ge \underline{a}_{n-1}(F) - \varepsilon,$$

where φ is the support function defined by (a.2). Going backwards from N to 0, we check that

$$u_n(\langle H, \Delta S\rangle + F) \ge \underline{a}_n(F) - (N-n)\varepsilon, \quad n=0,\dots,N.$$

Consequently, $\underline{V}_n(F) \geq \underline{a}_n(F)$ for any n.

It follows from [10; Lem. 4.3] that, for any n, there exists $Z_n^* \in \mathcal{D}_n$ such that

$$\mathsf{E}(Z_n^*(\Delta S_n, F_n + \underline{a}_n(F)) \,|\, \mathcal{F}_{n-1}) = (0, \underline{a}_{n-1}(F)).$$

This means, in particular, that $Z_n^* \in \mathcal{D}_n \cap \mathcal{M}_n$. Going backwards from N to 0, we check that $u_n(F; \mathcal{D} \cap \mathcal{M}) \leq \underline{a}_n(F)$ for any n.

(ii) If $H_n \in \underline{\mathcal{H}}_n(F)$ for any n, then

$$\varphi_{(0,\underline{a}_{n-1}(F))}(\underline{G}_{n-1}(F)) = \langle (H_n, 1), (0, \underline{a}_{n-1}(F)) \rangle = \underline{a}_{n-1}(F), \quad n = 1, \dots, N,$$

so that, by Lemma A.4,

$$\operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(\langle H_n, \Delta S_n \rangle + F_n + \underline{a}_n(F)) \,|\, \mathcal{F}_{n-1}) = \underline{a}_{n-1}(F), \quad n = 1, \dots, N,$$

and going backwards from N to 0, we check that $u_n(\langle H, \Delta S \rangle + F) = \underline{a}_n(F)$ for any n.

Take now $H \in \mathcal{H}$ such that (3.6) is satisfied. For any n, we have

$$\begin{aligned}
&\{H_n \in \beta(\underline{G}_{n-1}(F))\} \\
&= \Big\{\langle (H_n, 1), (0, \alpha(\underline{G}_{n-1}(F)) \rangle = \inf_{x \in \underline{G}_{n-1}(F)} \langle (H_n, 1), x \rangle \Big\} \\
&= \{\varphi_{(H_n,1)}(\underline{G}_{n-1}(F)) = \alpha(\underline{G}_{n-1}(F))\} \\
&= \Big\{\operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(\langle H_n, \Delta S_n \rangle + F_n + \underline{a}_n(F)) \,|\, \mathcal{F}_{n-1}) = \underline{a}_{n-1}(F)\Big\} \\
&= \Big\{\operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(\langle H_n, \Delta S_n \rangle + F_n + u_n(\langle H, \Delta S \rangle + F)) \,|\, \mathcal{F}_{n-1}) = \underline{a}_{n-1}(F)\Big\} \\
&= \{u_{n-1}(\langle H, \Delta S \rangle + F) = \underline{V}_{n-1}(F)\}.
\end{aligned}$$

The third equality here follows from Lemma A.4. Thus, $H_n \in \beta(\underline{G}_{n-1}(F))$ a.s.

Let now $\mathcal{D}$ be strictly positive. Let $H \in \mathcal{H}$ be such that (3.6) is satisfied for $n = 0$. We will prove that (3.6) is satisfied for any n going forwards from 0 to N. Suppose that (3.6) is true for $n-1$ and let us prove it for n. According to [10; Lem. 2.3], there exists $Z_n^* \in \mathcal{D}_n \cap \mathcal{M}_n$ such that

$$u_{n-1}(F; \mathcal{D} \cap \mathcal{M}) = \mathsf{E}(Z_n^*(F_n + u_n(F; \mathcal{D} \cap \mathcal{M})) \,|\, \mathcal{F}_{n-1}).$$

Due to the induction assumption,

$$\begin{aligned}
u_{n-1}(F; \mathcal{D} \cap \mathcal{M}) = \underline{V}_{n-1}(F) &= u_{n-1}(\langle H, \Delta S \rangle + F) \\
&\leq \mathsf{E}(Z_n^*(\langle H_n, \Delta S_n \rangle + F_n + u_n(\langle H, \Delta S \rangle + F)) \,|\, \mathcal{F}_{n-1}) \\
&= \mathsf{E}(Z_n^*(F_n + u_n(\langle H, \Delta S \rangle + F)) \,|\, \mathcal{F}_{n-1}).
\end{aligned}$$

As $Z_n^* > 0$ a.s. and $u_n(\langle H, \Delta S \rangle + F) \leq \underline{V}_n(F) = u_n(F; \mathcal{D} \cap \mathcal{M})$, we conclude that (3.6) is true for n. □

`Remarks`. (i) The representation $\underline{V}_n(F) = u_n(F; \mathcal{D} \cap \mathcal{M})$ remains valid also for the infinite-dimensional framework of the previous section.

(ii) According to [10; Prop. 2.2], there exist $Z_n^* \in \mathcal{D}_n \cap \mathcal{M}_n$ such that

$$\underline{V}_n(F) = \operatorname*{essinf}_{Z_k \in \mathcal{D}_k \cap \mathcal{M}_n} \mathsf{E}\bigg(\sum_{m=n+1}^{N} Z_{n+1} ... Z_m F_m \bigg| \mathcal{F}_n \bigg) = \mathsf{E}\bigg(\sum_{m=n+1}^{N} Z_{n+1}^* ... Z_m^* F_m \bigg| \mathcal{F}_n \bigg), \quad n = 0, ..., N.$$

If moreover $\mathcal{D}$ is probabilistic, then[3]

$$\underline{V}_n(F) = \operatorname*{essinf}_{\mathsf{Q}\in\widetilde{\mathcal{D}}} \mathsf{E}_{\mathsf{Q}}\biggl(\sum_{m=n+1}^{N} F_m \biggm| \mathcal{F}_n\biggr) = \mathsf{E}_{\mathsf{Q}_*}\biggl(\sum_{m=n+1}^{N} F_m \biggm| \mathcal{F}_n\biggr), \quad n = 0,\dots,N.$$

In particular, $\underline{V}_n(F)$ depends only on the cumulative cash flow remained after time n and does not depend on the timing of payments in this cash flow.

(iii) If each $\underline{\mathcal{H}}_n(F)$ is non-empty, then there exists a strategy $H \in \mathcal{H}$ such that $u_n(\langle h, \Delta S\rangle + F) = 0$ for any n. This means that the superreplication strategy does not depend on the initial date; its value at any time depends only on the residual part of the stream. This is very convenient from the practical point of view; otherwise, the method would have been inapplicable. This consistency property has already been pointed out by Leitner [21] in the continuous-time diffusion framework.

(iv) The strict positivity of $\mathcal{D}$ in Theorem 3.2 (ii) is essential. To see this, consider the example: $N = 2$, $\mathcal{F}_0 = \text{triv}$, $\mathcal{F}_1 = \sigma(\eta_1)$, $\mathcal{F}_2 = \sigma(\eta_1,\eta_2)$, $\mathcal{D}_n = \{Z \in L^0(\mathcal{F}_n) : 0 \le Z \le 2,\ \mathsf{E}(Z|\mathcal{F}_{n-1}) = 1\}$, $S_0 = S_1 = 0$, $S_2 = \eta_2$, $F_1 = 0$, and $F_2 = 10I(\eta_1 = 1)$. Here η_1, η_2 are independent and $\mathsf{P}(\eta_i = \pm 1) = 1/2$. Then $\underline{V}_0(F) = 0$ and $\underline{\mathcal{H}}_2(F) = \{0\}$. However, for $H = (H_1, H_2)$, where $H_1 = 0$, $H_2 = I(\eta_1 = 1)$, we have $u_0(\langle H, \Delta S\rangle + F) = 0$. □

3.2 Price Contribution

Consider the model of the previous subsection and let $F' = (F'_n)_{n=0,\dots,N}$ be another process such that $F'_n \in L^1(\overline{\mathcal{D}})$ for any n. From the financial point of view, F is the stream of cash flows corresponding to a large portfolio and F' is the stream of cash flows produced by an additional contract. Let us study the problem: how is the price process altered when F is replaced by $F+F'$? From the viewpoint of the next subsection, this problem is: how is the risk of a large portfolio altered after a contract has been added to it? Of course, the procedure for calculating the price for F is carried over to $F+F'$. However, if F corresponds to a huge portfolio, then recalculating the price is time consuming and it is desirable to have at least an approximate but fast method to estimate the price of $F+F'$.

The definition below was introduced in [10]. We use the notation

$$\operatorname*{argessmin}_{\xi\in A} \xi = \Bigl\{\xi \in A : \xi = \operatorname*{essinf}_{\xi'\in A} \xi'\Bigr\}.$$

Definition 3.3. The *extreme system* corresponding to a process X and a determining system $\mathcal{D}'$ is defined as $\mathcal{X}(X;\mathcal{D}') = (\mathcal{X}_n(X;\mathcal{D}'))_{n=1,\dots,N}$, where

$$\mathcal{X}_n(X;\mathcal{D}') = \operatorname*{argessmin}_{Z\in\mathcal{D}'_n} \mathsf{E}(Z u_n(X;\mathcal{D}')\,|\,\mathcal{F}_{n-1}).$$

If $X \in L^1(\overline{\mathcal{D}'})$, then $\mathcal{X}_n(X;\mathcal{D}') \ne \emptyset$ for any n (see [10; Sect. 4]). Theorem 3.2, combined with [10; Th. 6.2], yields

Corollary 3.4. *We have*

$$\text{(a.s.)} \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(\underline{V}_n(F+\varepsilon F') - \underline{V}_n(F)) = u_n(F';\mathcal{X}(F;\mathcal{D}\cap\mathcal{M})), \quad n = 0,\dots,N.$$

[3] Throughout the paper, the conditional expectations $\mathsf{E}_{\mathsf{Q}}(\xi|\mathcal{F}_n)$ are understood with the convention: $\mathsf{E}_{\mathsf{Q}}(\xi|\mathcal{F}_n) = 0$ on the set $\bigl\{\frac{d\mathsf{Q}|\mathcal{F}_n}{d\mathsf{P}|\mathcal{F}_n} = 0\bigr\}$.

The above expression might be called the *price contribution* of F' to F. Informally, the corollary says that if F' is small as compared to F, then

$$\underline{V}_n(F+F') - \underline{V}_n(F) \approx u_n(F'; \mathcal{X}(F; \mathcal{D} \cap \mathcal{M})), \quad n = 0, \dots, N. \tag{3.7}$$

`Remarks.` (i) Typically, $\mathcal{X}_n(F; \mathcal{D} \cap \mathcal{M})$ consists of a unique element Z_n (as an example, see Theorem 4.8). In this case one easily checks by the backward induction that

$$u_n(F'; \mathcal{X}(F; \mathcal{D} \cap \mathcal{M})) = \mathsf{E}\Bigg(\sum_{m=n+1}^{N} Z_{n+1} \dots Z_m F'_m \,\Bigg|\, \mathcal{F}_n\Bigg), \quad n = 0, \dots, N.$$

If moreover $\mathcal{D}$ is probabilistic, then

$$u_n(F'; \mathcal{X}(F; \mathcal{D} \cap \mathcal{M})) = \mathsf{E}_{\mathsf{Q}}\Bigg(\sum_{m=n+1}^{N} F'_m \,\Bigg|\, \mathcal{F}_n\Bigg), \quad n = 0, \dots, N, \tag{3.8}$$

where $\mathsf{Q} = Z_1 \dots Z_N \mathsf{P}$. The measure Q might be called the *extreme measure* corresponding to F. In view of (3.7), it might be regarded as the personal valuation measure of the agent possessing the portfolio F.[4]

(ii) Suppose that $F = F^1 + \dots + F^K$, where $F^k_n \in L^1(\overline{\mathcal{D}})$ for any n, k. Suppose that each $\mathcal{X}_n(F; \mathcal{D} \cap \mathcal{M})$ consists of a unique element Z_n. Then, according to the previous remark,

$$\sum_{k=1}^{K} u_n(F^k; \mathcal{D} \cap \mathcal{M}) = \sum_{k=1}^{K} \mathsf{E}\Bigg(\sum_{m=n+1}^{N} Z_{n+1} \dots Z_m F^k_m \,\Bigg|\, \mathcal{F}_n\Bigg) = u_n(F; \mathcal{D} \cap \mathcal{M}) = \underline{V}_n(F).$$

So, the sum of price contributions of the components of a portfolio typically equals the price of the portfolio. □

3.3 Risk Measurement and Management

Consider the model of Subsection 3.1 and assume that $\mathcal{F}_0$ is trivial.[5] Let $\gamma \in \mathbb{R}^d$. From the financial point of view, we have a large portfolio consisting of two parts: the liquid part consists of γ^i assets of the i-th type, $i = 1, \dots, d$ and the illiquid part produces a stream of cash flows F. The holding γ is already fixed for the time period $[0, 1]$, and the liquid part can be rebalanced at times $1, 2, \dots$;[6] there is no trading in the illiquid assets, so that the illiquid part cannot be changed. The time 0 is the current time and the time N is chosen in such a way that there are no cash flows in the illiquid part after this time.

The set of strategies available to the portfolio holder is the set $\mathcal{H}(\gamma)$ of d-dimensional predictable processes H such that $H_1 = \gamma$.

---

[4] It serves as the dynamic risk-based analog of the classical valuation measure $cU'(W)$, where U is an agent's utility function, W is his/her wealth, and c is a normalizing constant.

[5] This assumption is imposed because we want to make a link to classical static (not conditional) risk measures. However, the results are easily carried over to a general $\mathcal{F}_0$.

[6] Thus, the unit time period serves as the minimal time needed to get rid of the liquid assets (for example, one day).

Definition 3.5. The *market-adjusted utility* of the portfolio (γ, F) is

$$u^m(\gamma, F) = \sup_{H \in \mathcal{H}(\gamma)} u_0(\langle H, \Delta S\rangle + F).$$

The *market-adjusted risk* is $\rho^m(\gamma, F) = -u^m(\gamma, F)$.[7]

Financially, the problem of finding sup here is the problem of risk measurement. Mathematically, it coincides with the pricing problem. Financially, the problem of finding the optimal H here is the problem of risk management. Mathematically, it coincides with the hedging problem.

Consider the determining system $\mathcal{D}^m$ given by: $\mathcal{D}_1^m = \mathcal{D}_1$ and $\mathcal{D}_n^m = \mathcal{D}_n \cap \mathcal{M}_n$ for $n = 2, \dots, N$. Theorem 3.2 yields

Corollary 3.6. *We have* $u^m(\gamma, F) = u_0(\langle \gamma, \Delta S\rangle + F; \mathcal{D}^m)$.

Let now (γ', F') be another portfolio with $F'_n \in L^1(\overline{\mathcal{D}})$. Introduce the notation

$$\mathcal{X}_n^m(F) = \begin{cases} \operatorname*{argmin}_{Z \in \mathcal{D}_1} \mathsf{E} Z(\langle \gamma, \Delta S_1\rangle + F_1 + u_1(F; \mathcal{D} \cap \mathcal{M})), & n = 1, \\ \mathcal{X}_n(F; \mathcal{D} \cap \mathcal{M}), & n = 2, \dots, N. \end{cases}$$

It is easy to check that $\mathcal{X}_n(F; \mathcal{D}^m) = \mathcal{X}_n^m(F)$ for any n. Corollary 3.6, combined with [10; Th. 6.2], yields the following expression for the *market-adjusted utility contribution* of (γ', F') to (γ, F):

Corollary 3.7. *We have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}[u^m(\langle \gamma, F\rangle + \varepsilon\langle \gamma', F'\rangle) - u^m(\langle \gamma, F\rangle)] = \inf_{Z_n \in \mathcal{X}_n^m(F)} \mathsf{E}\Big[Z_1\langle \gamma', \Delta S_1\rangle + \sum_{n=1}^N Z_1 \dots Z_n F'_n\Big].$$

Let us now suppose that the portfolio consists of several subportfolios, i.e. $(\gamma, F) = (\gamma^1, F^1) + \dots + (\gamma^K, F^K)$, where $\gamma^k \in \mathbb{R}^d$, $F_n^k \in L^1(\overline{\mathcal{D}})$. Let us study how the risk of the whole portfolio is divided between the subportfolios. The following definition is a reformulation for our case of the one from Delbaen [14; Sect. 9].

Definition 3.8. A *market-adjusted utility allocation* between $(\gamma^1, F^1), \dots, (\gamma^K, F^K)$ is a collection of real numbers $x^1, \dots, x^K$ such that

$$\sum_{k=1}^K x^k = u^m(\gamma, F),$$
$$\sum_{k=1}^K h^k x^k \geq u^m\Big(\sum_{k=1}^K h^k(\gamma^k, F^k)\Big) \quad \forall h \in \mathbb{R}_+^K.$$

A *market-adjusted capital allocation* is $-x^1, \dots, -x^K$.

[7] The reader might think that it would be more natural to consider here the optimization over the whole set $\mathcal{H}$, not only $\mathcal{H}(\gamma)$. The choice we make reflects the following: at time 0 an agent already has a fixed holding in liquid assets; this holding is chosen not by risk minimizing considerations, but by some exogenous considerations. The market-adjusted risk is thus the minimal risk that can be achieved by rebalancing the portfolio in the future.

Corollary 3.6, combined with [10; Lem. 4.4] and [10; Th. 5.2], yields

Corollary 3.9. *A collection* $x^1,\dots,x^K$ *is a market-adjusted utility allocation between* $(\gamma^1,F^1),\dots,(\gamma^K,F^K)$ *if and only if there exist* $Z_n\in\mathcal{X}_n^m(F)$ *such that*

$$x^k=\mathsf{E}Z_1\langle\gamma^k,\Delta S_1\rangle+\mathsf{E}\sum_{n=1}^N Z_1\dots Z_nF_n^k,\quad k=1,\dots,K.$$

Remarks. (i) It follows from [10; Sect. 4] that $\mathcal{X}_n^m(F)$ is non-empty for any n, so that a market-adjusted utility allocation exists.

(ii) Typically, each $\mathcal{X}_n^m(F)$ is a singleton (as an example, see Theorem 4.8). In this case the market-adjusted utility allocation is unique and its components are exactly the market-adjusted utility contributions of (γ^k,F^k) to (γ,F). If moreover $\mathcal{D}$ is probabilistic, then $x^k=\mathsf{E}_\mathsf{Q}\big(\langle\gamma^k,\Delta S_1\rangle+\sum_n F_n^k\big)$, where $\mathsf{Q}=Z_1\dots Z_N\mathsf{P}$. The measure Q might be called the *market-adjusted extreme measure* of an agent possessing the portfolio (γ,F). □

Finally, let us find out how the market-adjusted risk is divided between the liquid and the illiquid parts of the portfolio. Thus, we should consider $(\gamma^1,F^1)=(\gamma,0)$, $(\gamma^2,F^2)=(0,F)$. Then, according to Corollary 3.9, any market-adjusted utility allocation has the form

$$x^1=\mathsf{E}Z_1\langle\gamma,\Delta S_1\rangle,\qquad x^2=\mathsf{E}\sum_{n=1}^N Z_1\dots Z_nF_n=\mathsf{E}Z_1(F_1+u_1(F;\mathcal{D}\cap\mathcal{M})),$$

where $Z_n\in\mathcal{X}_n^m(F)$. (Note that here x_1,x_2 depend only on Z_1 and do not depend on $Z_2,\dots,Z_N$.) In particular,

$$\rho^m(\gamma,F)=-x^1-x^2\le\rho^s(\langle\gamma,\Delta S_1\rangle)+\rho_0(F;\mathcal{D}\cap\mathcal{M}).$$

Here $\rho^s(X)=-\inf_{Z\in\mathcal{D}_1}\mathsf{E}ZX$. If $\mathsf{E}Z=1$ for any $Z\in\mathcal{D}_1$, then ρ^s is a static coherent risk measure. Thus, we have "proved" (1.1).

4 Pricing and Hedging in the Markov Model

4.1 A- and B-Operators

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space. Recall that a basic static risk measure *Tail V@R* is defined as $\rho_\lambda=-u_\lambda$, where

$$u_\lambda(X)=\inf_{Z\in\mathcal{D}_\lambda}\mathsf{E}ZX,\quad X\in L^0.$$

Here $\lambda\in(0,1]$ and

$$\mathcal{D}_\lambda=\{Z:0\le Z\le\lambda^{-1},\ \mathsf{E}Z=1\}.\tag{4.1}$$

The expectation $\mathsf{E}ZX$ is understood as $\mathsf{E}ZX^+-\mathsf{E}ZX^-$ with the convention $\infty-\infty=-\infty$.

Consider the functional

$$u_\mu(X)=\int_{(0,1]}\rho_\lambda(X)\mu(d\lambda),\quad X\in L^0,$$

where μ is a positive measure on $(0,1]$ with $\mu((0,1])\le 1$. The integral $\int_{(0,1]}f(x)\mu(dx)$ is understood as $\int_{(0,1]}f^+(x)\mu(dx)-\int_{(0,1]}f^-(x)\mu(dx)$ with the convention $\infty-\infty=-\infty$. For a probabilistic μ, $\rho_\mu=-u_\mu$ is a static risk measure termed *Weighted V@R*. According to [9; Th. 4.6], u_μ admits a representation

$$u_\mu(X)=\inf_{Z\in\mathcal{D}_\mu}\mathsf{E}ZX,\quad X\in L^0, \tag{4.2}$$

where

$$\mathcal{D}_\mu=\{Z:Z\ge 0,\ \mathsf{E}Z=\mu((0,1]),\ \text{and } \mathsf{E}(Z-x)^+\le\Phi_\mu(x)\ \forall x\in\mathbb{R}_+\}, \tag{4.3}$$

$$\Phi_\mu(x)=\sup_{y\in[0,1]}\biggl[\int_0^y\int_{[z,1]}\lambda^{-1}\mu(d\lambda)dz-xy\biggr],\quad x\in\mathbb{R}_+. \tag{4.4}$$

Let μ be a positive measure on $(0,1]$ with $\mu((0,1])\le 1$. Let $d\in\mathbb{N}$, $m\in\mathbb{Z}_+$, and D_0, D_1 be measurable subsets of $\mathbb{R}^{d+m}$. We will denote points of $\mathbb{R}^{d+m}$ as (x,y), where $x\in\mathbb{R}^d$, $y\in\mathbb{R}^m$; by X and Y we will denote the projections of $\mathbb{R}^{d+m}$ on $\mathbb{R}^d$ and $\mathbb{R}^m$. Let $(P(x,y))_{(x,y)\in D_0}$ be a family of probability measures on D_1. We assume that P is weakly continuous in (x,y) and, for any $(x,y)\in D_0$, there exists $\varepsilon>0$ such that

$$\inf_{(x',y')\in B_\varepsilon(x,y)\cap D_0}u_{\mu,P(x',y')}(-|X|I(|X|>k))\xrightarrow[k\to\infty]{}0, \tag{4.5}$$

where $B_\varepsilon(x,y)$ is the ball of radius ε centered at (x,y) and $u_{\mu,P(x',y')}$ is defined by (4.2) with $\Omega=D_1$, $\mathsf{P}=P(x',y')$. We also assume that

$$u_{\mu,P(x,y)}(\langle h,X-x\rangle)<0\ \ \forall h\in\mathbb{R}^d\setminus\{0\},\ (x,y)\in D_0. \tag{4.6}$$

Introduce the notation

$$\begin{aligned}\mathcal{L}(D)=\{f\in C(D):\exists a_1,a_2,b_1,b_2\in\mathbb{R},\ h_1,h_2\in\mathbb{R}^d:\\ a_1\langle h_1,x\rangle+b_1\le f(x,y)\le a_2\langle h_2,x\rangle+b_2,\ (x,y)\in D\},\end{aligned}$$

where $C(D)$ is the space of the continuous functions $D\to\mathbb{R}$.

Definition 4.1. The A- and B-*operators* are defined as:

$$A_{\mu,P}f(x,y)=\sup_{h\in\mathbb{R}^d}u_{\mu,P(x,y)}(\langle h,X-x\rangle+f(X,Y)),\quad (x,y)\in D_0,\ f\in\mathcal{L}(D_1),$$

$$B_{\mu,P}f(x,y)=\operatorname*{argmax}_{h\in\mathbb{R}^d}u_{\mu,P(x,y)}(\langle h,X-x\rangle+f(X,Y)),\quad (x,y)\in D_0,\ f\in\mathcal{L}(D_1).$$

We will say that a set $S\subseteq\mathbb{R}^{d+m}$ is *strongly connected* if $S\cap\{x\in\mathbb{R}^d:a_1\le\langle h,x\rangle\le a_2\}$ is connected for any $a_1,a_2,\in\mathbb{R}$, $h\in\mathbb{R}^d$. Below "supp" denotes the support and "Law" denotes the distribution.

Theorem 4.2. **(i)** *For any* $f\in\mathcal{L}(D_1)$, *we have* $A_{\mu,P}f\in\mathcal{L}(D_0)$.

(ii) *For any* $f\in\mathcal{L}(D_1)$, $(x,y)\in D_0$, *we have* $B_{\mu,P}f(x,y)\ne\emptyset$.

(iii) *If* $\mu((0,1))>0$, $f\in\mathcal{L}(D_1)$, *and, for some* $(x,y)\in D_0$, $\operatorname{supp}P(x,y)\cap D_1$ *is strongly connected and* $\mathsf{Law}_{P(x,y)}\langle h,X\rangle$ *is continuous for any* $h\in\mathbb{R}^d\setminus\{0\}$, *then* $B_{\mu,P}f(x,y)$ *is a singleton.*

Proof. **(i)** Fix $f \in \mathcal{L}(D_1)$. Denote

$$X(n) = \begin{cases} X, & |X| \le n, \\ nX/|X|, & |X| > n. \end{cases}$$

It follows from (4.5) that, for any $h_1 \in \mathbb{R}^d$, $h_2 \in \mathbb{R}$,

$$u_{\mu,P(x,y)}(\langle h_1, X(n) - x\rangle + h_2 f(X(n), Y)) \xrightarrow[n\to\infty]{} u_{\mu,P(x,y)}(\langle h_1, X - x\rangle + h_2 f(X, Y))$$

locally uniformly in (x, y). Furthermore, the map

$$D_0 \ni (x, y) \longmapsto \mathsf{Law}_{P(x,y)}(\langle h_1, X(n) - x\rangle + h_2 f(X(n), Y))$$

is weakly continuous and, for any $(x, y) \in D_1$ there exist $\varepsilon > 0$ and a compact interval $I \subset \mathbb{R}$ such that the restriction of this map to $B_\varepsilon(x, y)$ takes values in the space $\mathfrak{M}(I)$ of probability measures concentrated on I.

It is well known that $u_\lambda(X)$ depends only on the distribution of X (see, for example, [9; Prop. 2.7]), so that there exists a map $\widetilde{u}_\lambda$ defined on distributions such that $u_\lambda(X) = \widetilde{u}_\lambda(\mathsf{Law}\, X)$, $X \in L^0$. The same is true for u_μ. For any compact interval $I \subset \mathbb{R}$, the map $\mathfrak{M}(I) \ni Q \mapsto u_\lambda(Q)$ is weakly continuous as can be seen from the explicit representation of $\widetilde{u}_\lambda$; see [9; Prop 2.7]. Approximating μ by finite sums of the form $\sum \alpha_k \delta_{\lambda_k}$ (δ_λ denotes the delta-mass concentrated at λ), we see that the map $\mathfrak{M}(I) \ni Q \mapsto \widetilde{u}_\mu(Q)$ is weakly continuous. Thus, we have proved for any $h_1 \in \mathbb{R}^d$, $h_2 \in \mathbb{R}$ the continuity of the map

$$D_0 \ni (x, y) \longmapsto u_{\mu,P(x,y)}(\langle h_1, X - x\rangle + h_2 f(X, Y)).$$

Consider the map $D_0 \ni (x, y) \to \mathcal{C}$ defined as

$$G(x, y) = \mathrm{cl}\{\mathsf{E}_{P(x,y)} Z(X - x, f(X, Y)) : Z \in \mathcal{D}_{\mu,P(x,y)}\}, \quad (x, y) \in D_0, \tag{4.7}$$

where "cl" denotes the closure and $\mathcal{D}_{\mu,P(x,y)}$ is defined by (4.3) with $\Omega = D_1$, $\mathsf{P} = P(x, y)$. The result proved above says that, for any $h \in \mathbb{R}^{d+1}$, the map $D_0 \ni (x, y) \mapsto \varphi_h(G(x, y))$ is continuous, where φ is the support function given by (a.2). From this it follows that the map $D_0 \ni (x, y) \mapsto G(x, y)$ is continuous in the Hausdorff metrics (a.1). Due to (4.6), $G(x, y) \in \mathcal{C}_0$ for any $(x, y) \in D_0$, where $\mathcal{C}_0$ is the set of elements $C \in \mathcal{C}$ such that 0 belongs to the interior of $\mathrm{pr}_{\mathbb{R}^d} C$. It is seen from the line

$$\begin{aligned} u_{\mu,P(x,y)}(\langle h, X - x\rangle + f(X, Y)) &= \inf_{Z \in \mathcal{D}_{\mu,P(x,y)}} \langle (h, 1), \mathsf{E}_{P(x,y)} Z(X - x, f(X, Y))\rangle \\ &= \inf_{z \in G(x,y)} \langle (h, 1), z\rangle, \quad h \in \mathbb{R}^d,\ (x, y) \in D_0 \end{aligned} \tag{4.8}$$

that

$$A_{\mu,P} f(x, y) = \alpha(G(x, y)), \quad (x, y) \in D_0, \tag{4.9}$$

where α is given by (3.1). Clearly, the restriction of α to $\mathcal{C}_0$ is continuous in the Hausdorff metrics. As a result, the map $D_0 \ni (x, y) \mapsto A_{\mu,P} f(x, y)$ is continuous.

Find $a, b \in \mathbb{R}$, $h \in \mathbb{R}^d$ such that $f(x, y) \le a\langle h, x\rangle + b$ for any $(x, y) \in D_1$. Then

$$\begin{aligned} A_{\mu,P} f(x, y) &\le \sup_{h' \in \mathbb{R}^d} u_{\mu,P(x,y)}(\langle h', X - x\rangle + a\langle h, X\rangle + b) \\ &= \sup_{h' \in \mathbb{R}^d} u_{\mu,P(x,y)}(\langle h' + ah, X - x\rangle) + \mu((0, 1])(a\langle h, x\rangle + b) \\ &= \mu((0, 1])(a\langle h, x\rangle + b), \quad (x, y) \in D_0. \end{aligned}$$

The second equality here follows from (4.6). A similar estimate holds below. Thus, $A_{\mu,P}f \in \mathcal{L}(D_0)$.

(ii) It is seen from (4.8) that

$$B_{\mu,P}f(x,y) = \beta(G(x,y)), \quad (x,y) \in D_0, \tag{4.10}$$

where β is given by (3.2). As 0 belongs to the interior of $\mathrm{pr}_{\mathbb{R}^d} G(x,y)$, we get the desired statement.

(iii) Suppose that there exist $h_1 \neq h_2 \in B_{\mu,P}f(x,y)$. Without loss of generality, $h_1 = 0$. Denote $P(x,y)$ by P. As $\lim_{x\to\infty}\Phi_\mu(x) = 0$, the set $\mathcal{D}_{\mu,P}$ is P-uniformly integrable. By the Dunford-Pettis criterion, $\mathcal{D}_{\mu,P}$ is P-relatively weakly compact. As $\mathcal{D}_{\mu,P}$ is $L^1(P)$-closed and convex, it is P-weakly closed by the Hahn-Banach theorem. Thus, $\mathcal{D}_{\mu,P}$ is P-weakly compact. It follows from (4.5) combined with [7; Prop. 2.6] that $X \in L^1(\mathcal{D}_{\mu,P})$. Hence, the map $\mathcal{D}_{\mu,P} \ni Z \mapsto \mathsf{E}_P Z(X - x, f(X,Y))$ is P-weakly continuous. Consequently, there exists $Z_* \in \mathcal{D}_\mu$ such that $\mathsf{E}_P Z_*(X-x) = 0$ and $\mathsf{E}_P Z_* f(X,Y) = A_{\mu,P}f(x,y)$ (here we recall (4.9)). Then

$$Z_* \in \operatorname*{argmin}_{Z\in\mathcal{D}_{\mu,P}} \mathsf{E}_P Z(\langle h_i, X-x\rangle + f(X,Y)), \quad i=1,2.$$

Let us make in $\mathbb{R}^d$ a change of coordinates, so that in the new coordinates the vector h_2 is written as $(1,0,\dots,0)$. Then any point (x',y') of $\mathbb{R}^{d+m}$ can be written as (u,v), where $u \in \mathbb{R}$ is the first coordinate of x' in the new system and $v \in \mathbb{R}^{d+m-1}$ is the vector that consists of the remaining $d-1$ coordinates of x' in the new system and the vector y'. Define $g(u,v) = f(u,v)$, $\widetilde{g}(u,v) = u + f(u,v)$, where $f(u,v)$ is the function f rewritten in the new coordinates. Then $\mathsf{E}Z_*(U - x_0) = 0$ and

$$Z_* \in \operatorname*{argmin}_{Z\in\mathcal{D}_{\mu,P}} \mathsf{E}_P Z g(U,V) \cap \operatorname*{argmin}_{Z\in\mathcal{D}_{\mu,P}} \mathsf{E}_P Z\widetilde{g}(U,V),$$

where x_0 is the first coordinate of x in the new system and U, V denote the projections of $\mathbb{R}^{d+m}$ on $\mathbb{R}$, $\mathbb{R}^{d+m-1}$. Furthermore, it follows from (4.6) that $u_{\mu,P}(U-x_0) < 0 < -u_{\mu,P}(-U+x_0)$.

According to [9; Th. 4.4], Z_* can be represented as $Z_* = \int_{(0,1]} Z_\lambda \mu(d\lambda)$, where $Z(\lambda,\omega)$ is jointly measurable and $Z_\lambda \in \mathcal{D}_{\lambda,P}$ for any $\lambda \in (0,1]$, where $\mathcal{D}_{\lambda,P}$ is given by (4.1) with $\Omega = D_1$, $\mathsf{P} = P$. It is seen from the line

$$\begin{aligned} u_{\mu,P}(g(U,V)) = \mathsf{E}_P Z_* g(U,V) &= \int_{(0,1]} \mathsf{E}_P Z_\lambda g(U,V)\mu(d\lambda) \\ &\geq \int_{(0,1]} u_{\lambda,P}(g(U,V))\mu(d\lambda) = u_{\mu,P}(g(U,V)) \end{aligned}$$

that $Z_\lambda \in \operatorname{argmin}_{Z\in\mathcal{D}_{\lambda,P}} \mathsf{E}_P Z g(U,V)$ for μ-a.e. λ. The same is true for $\widetilde{g}(U,V)$.

Fix $\lambda \in (0,1)$ such that

$$Z_\lambda \in \operatorname*{argmin}_{Z\in\mathcal{D}_{\lambda,P}} \mathsf{E}_P Z g(U,V) \cap \operatorname*{argmin}_{Z\in\mathcal{D}_{\lambda,P}} \mathsf{E}_P Z\widetilde{g}(U,V). \tag{4.11}$$

Without loss of generality, 0 is a λ-quantile of $\mathsf{Law}_P\, g(U,V)$. Below we will use the notation: $\overline{S} = \operatorname{supp} P \cap D_1$, $S_{(a,b)} = \{(u,v) \in \overline{S} : u \in (a,b)\}$, $l = \inf\{u : \exists v : (u,v) \in \overline{S}\}$,

$r = \sup\{u : \exists v : (u,v) \in \overline{S}\}$, $S = \overline{S}_{(l,r)}$,

$$\begin{aligned}
C^1 &= \{(u,v) \in \mathbb{R}^{d+m} : \exists \varepsilon > 0 : Z_\lambda = \lambda^{-1} \text{ a.e. on } B_\varepsilon(u,v) \cap D_1\},\\
C^0 &= \{(u,v) \in \mathbb{R}^{d+m} : \exists \varepsilon > 0 : Z_\lambda = 0 \text{ a.e. on } B_\varepsilon(u,v) \cap D_1\},\\
E^+ &= \{u \in (l,r) : \exists v : (u,v) \in S \text{ and } g(u,v) > 0\},\\
E^0 &= \{u \in (l,r) : \exists v : (u,v) \in S \text{ and } g(u,v) = 0\},\\
E^- &= \{u \in (l,r) : \exists v : (u,v) \in S \text{ and } g(u,v) < 0\}.
\end{aligned}$$

Let us consider several cases.

Case 1. Suppose that $g = 0$ P-a.s. Then $\widetilde{g}(u,v) = u$ P-a.s. Due to the inclusion $Z_* \in \operatorname{argmin}_{Z \in \mathcal{D}_{\mu,P}} \mathsf{E}_P Z\widetilde{g}(U,V)$, we get $\mathsf{E}_P Z_*(U - x_0) = u_{\mu,P}(U - x_0) < 0$, which contradicts the choice of Z_*. So, this case is excluded.

Case 2. Suppose that $g \le 0$ P-a.s.

Case 2.1. Suppose that $\sup\{u : u \in E^0\} = r$ and $\sup\{u : u \in E^-\} = r$. Take $(u,v) \in S$ with $u < r$. Using Lemma 4.4, we can find $(u',v') \in S$ such that $u < u' < r$, $g(u',v') < 0$, and $\widetilde{g}(u',v') > \widetilde{g}(u,v)$. Applying Lemma 4.3 to $\xi = g(U,V)$, we get $(u',v') \in C^1$. Applying Lemma 4.3 to $\xi = \widetilde{g}(U,V)$, we get $(u,v) \in C^1$. As (u,v) is arbitrary, we see that $Z_\lambda = \lambda^{-1}$ a.s. But this contradicts the property $\lambda \in (0,1)$. So, this case is excluded.

Case 2.2. Suppose that $\sup\{u : u \in E^0\} < r$. Note that $\{u : u \in E^0\} \ne \emptyset$ since otherwise $g < 0$ P-a.s., so that 0 cannot be a λ-quantile of $g(U,V)$. Denote $a = \sup\{u : u \in E^0\}$. Applying Lemma 4.3 to $\xi = g(U,V)$, we get $S_{(a,r)} \subseteq C^1$. Take $(u,v) \in S$ with $u < a$. Using Lemma 4.4, we can find $(u',v') \in S$ such that $u' > a$ (then $g(u',v') < 0$) and $\widetilde{g}(u',v') > \widetilde{g}(u,v)$. Applying Lemma 4.3 to $\xi = \widetilde{g}(U,V)$, we get $(u,v) \in C^1$. As (u,v) is arbitrary, we get $S_{(l,a)} \subseteq C^1$. Furthermore, $\mathsf{Law}_P U$ is continuous, so that $P(U = a) = 0$. Thus, $Z_\lambda = \lambda^{-1}$ a.s. So, this case is excluded.

Case 2.3. Suppose that $\sup\{u : u \in E^-\} < r$. Note that $\{u : u \in E^-\} \ne \emptyset$ since otherwise $g = 0$ P-a.s. Denote $a = \sup\{u : u \in E^-\}$. Take $(u,v) \in S$ with $u < a$. Using Lemma 4.4, we can find $(u',v') \in S$ such that $u < u' < a$, $g(u',v') < 0$, and $\widetilde{g}(u',v') > \widetilde{g}(u,v)$. Applying Lemma 4.3 twice, we get $(u',v') \in C^1$. Thus, $S_{(l,a)} \subseteq C^1$. Furthermore, on $S_{(a,r)}$, $g = 0$, so that $\widetilde{g}(u,v) = u$. Applying Lemma 4.3 to $\xi = \widetilde{g}(U,V)$, we see that there exists $b \in [a,r)$ such that $S_{(l,b)} \subseteq C^1$ and $S_{(b,r)} \subseteq C^0$. Applying Lemma 4.3 to $\xi = U$, we get $Z_\lambda \in \operatorname{argmin}_{Z \in \mathcal{D}_{\lambda,P}} \mathsf{E}_P ZU$.

Case 3. Suppose that $g \ge 0$ P-a.s. Arguing in the same way as in Case 2, we conclude that there exists $b \in (l,r)$ such that $S_{(l,b)} \subseteq C^1$, $S_{(b,r)} \subseteq C^0$, $\widetilde{g}(u,v) = u$ on $S_{(l,b)}$, and $Z_\lambda \in \operatorname{argmin}_{Z \in \mathcal{D}_{\lambda,P}} \mathsf{E}_P ZU$.

Case 4. Suppose that g takes both strictly positive and strictly negative values on S. Make the following observation. If $(u,v) \in S$ is such that $g(u,v) > 0$, then (keeping in mind that $P(U = u) = 0$) we can find a sequence $(u_n,v_n) \in S$ such that $u_n \ne u$ and $(u_n,v_n) \to (u,v)$. Thus, E^+ has no isolated points. The same is true for E^-.

Case 4.1. Suppose that there exist $u_1 \in E^+$, $u_2 \in E^-$ such that $u_1 < u_2$.

Case 4.1.1. Suppose that there exists $u_0 \in E^0$ such that $u_0 < u_1$. According to the reasoning at the beginning of Case 4, we can find $u_1', u_1'' \in E^+$ such that $u_0 < u_1' < u_1'' < u_2$. Using Lemma 4.4, we can find $(u_3,v_3) \in S$, $(u_4,v_4) \in S$ such that $u_3 \in [u_0,u_1']$, $u_4 \in [u_1'',u_2]$, $g(u_3,v_3) > 0$, $g(u_4,v_4) < 0$, and $\widetilde{g}(u_3,v_3) < \widetilde{g}(u_4,v_4)$. Applying Lemma 4.3 to $\xi = g(U,V)$, we get $(u_3,v_3) \in C^0$, $(u_4,v_4) \in C^1$. Applying Lemma 4.3 to $\xi = \widetilde{g}(U,V)$, we get a contradiction. So, this case is excluded.

Case 4.1.2. Suppose that there exists $u_3 \in E^0$ such that $u_3 > u_2$. Arguing in the same way as in the previous case, we get a contradiction. So, this case is excluded.

Case 4.1.3. Suppose that there exist $a_1 \le a_2 \in (l,r)$ such that $g > 0$ on $S_{(l,a_1)}$, $g = 0$ on $S_{(a_1,a_2)}$, and $g < 0$ on $S_{(a_2,r)}$.

Case 4.1.3.1. Suppose that $a_1 < a_2$. Using Lemma 4.4, we can find $(u_1,v_1) \in S$, $(u_2,v_2) \in S$ such that $u_1 < a_1$, $u_2 > a_2$ (then $g(u_1,v_1) > 0$, $g(u_2,v_2) < 0$), and $\widetilde{g}(u_1,v_1) < \widetilde{g}(u_2,v_2)$. Applying Lemma 4.3 twice, we get a contradiction. So, this case is excluded.

Case 4.1.3.2. Suppose that $a_1 = a_2$. Applying Lemma 4.3 to $\xi = g(U,V)$, we get $S_{(l,a_1)} \subseteq C^0$ and $S_{(a_1,b)} \subseteq C^1$. Applying Lemma 4.3 to $\xi = U$, we get $Z_\lambda \in \operatorname{argmax}_{Z\in\mathcal{D}_{\lambda,P}} \mathsf{E}_P ZU$.

Case 4.2. Suppose that there exists $a \in (l,r)$ such that $g \le 0$ on $S_{(l,a)}$ and $g \ge 0$ on $S_{(a,r)}$. Denote $a_1 = \sup\{u \le a : u \in E^-\}$, $a_2 = \inf\{u \ge a : u \in E^+\}$. Take $(u,v) \in S$ with $u < a_1$. Using Lemma 4.4, we can find $(u',v') \in S$ such that $u < u' < a$, $g(u',v') < 0$, and $\widetilde{g}(u',v') > \widetilde{g}(u,v)$. Applying Lemma 4.3 twice, we get $(u,v) \in C^1$. Thus, $S_{(l,a_1)} \subseteq C^1$. Similarly, $S_{(a_2,r)} \subseteq C^0$. Furthermore, on $S_{(a_1,a_2)}$, $g = 0$ and $\widetilde{g}(u,v) = u$. Applying Lemma 4.3 to $\xi = \widetilde{g}(U,V)$, we see that there exists $b \in [a_1,a_2]$ such that $S_{(l,b)} \subseteq C^1$ and $S_{(b,r)} \subseteq C^0$. Applying Lemma 4.3 to $\xi = U$, we get $Z_\lambda \in \operatorname{argmin}_{Z\in\mathcal{D}_{\lambda,P}} \mathsf{E}_P ZU$.

Thus, we have considered all the possible cases and excluded them, except for cases 2.3, 3, 4.1.3.2, and 4.2. Let us now exclude them also. Suppose that

$$\mu\big(\lambda \in (0,1) : u_{\lambda,P}(U) < \mathsf{E}_P Z_\lambda U < -u_{\lambda,P}(-U)\big) > 0. \tag{4.12}$$

Then we can choose $\lambda \in (0,1)$ satisfying (4.11) and the inequality $u_{\lambda,P}(U) < \mathsf{E}_P Z_\lambda U < -u_{\lambda,P}(-U)$. Then cases 2.3, 3, 4.1.3.2, and 4.2 are excluded, and we obtain a contradiction with the existence of h_1, h_2.

Now, suppose that (4.12) is violated. In view of the inequality $u_{\lambda,P}(U) \le \mathsf{E}_P Z_\lambda U \le -u_\lambda(-U)$, this means that, for μ-a.e. $\lambda \in (0,1)$, we have either $\mathsf{E}_P Z_\lambda U = u_{\lambda,P}(U)$ or $\mathsf{E}_P Z_\lambda U = -u_{\lambda,P}(-U)$. If case 2.3, 3, or 4.2 is satisfied for some λ, then one can see (with the help of Lemma 4.3) from the resulting structure of $\widetilde{g}$ that

$$\operatorname*{argmin}_{Z\in\mathcal{D}_{\lambda,P}} \mathsf{E}_P Z\widetilde{g}(U,V) \cap \operatorname*{argmax}_{Z\in\mathcal{D}_{\lambda,P}} \mathsf{E}_P ZU = \emptyset \quad \forall \lambda \in (0,1).$$

This implies that $\mathsf{E}_P Z_\lambda U = u_{\lambda,P}(U)$ for μ-a.e. $\lambda \in (0,1)$. Furthermore, for $\lambda = 1$, $\mathcal{D}_{\lambda,P} = \{1\}$, so that $\mathsf{E}_P Z_1 U = u_{1,P}(U)$. Thus,

$$\mathsf{E}_P Z_*(U - x_0) = \int_{(0,1]} \mathsf{E}_P Z_\lambda (U - x_0)\mu(d\lambda) = \int_{(0,1]} u_{\lambda,P}(U - x_0)\mu(d\lambda) = u_{\mu,P}(U - x_0) < 0.$$

On the other hand, Z_* has been chosen so that $\mathsf{E}_P Z_*(U - x_0) = 0$. Thus, we arrive at a contradiction, which shows that cases 2.3, 3, and 4.2 cannot be realized. In a similar way, we exclude case 4.1.3.2. As a result, we arrive at a contradiction with the existence of h_1, h_2. □

Lemma 4.3. *Let ξ be an integrable random variable and $\lambda \in (0,1]$. Then an element $Z \in \mathcal{D}_\lambda$ belongs to $\operatorname{argmin}_{Z\in\mathcal{D}_\lambda} \mathsf{E} Z\xi$ if and only if $Z = \lambda^{-1}$ a.e. on $\{\xi < q_\lambda\}$ and $Z = 0$ a.e. on $\{\xi = q_\lambda\}$, where q_λ is a λ-quantile of ξ.*

`Proof`. Let Z_* be an element of the described form and Z be an arbitrary element of $\mathcal{D}_\lambda$. Let us prove that $\mathsf{E} Z_*\xi \le \mathsf{E} Z\xi$. Without loss of generality, we can assume that $q_\lambda = 0$. Then

$$Z\xi - Z^*\xi = (Z - \lambda^{-1})\xi I(\xi < 0) + Z\xi I(\xi > 0) \ge 0.$$

Furthermore, the a.s. equality here is possible only if Z also has the described form. □

Lemma 4.4. *Let $S \subseteq \mathbb{R}^{d+m}$ be such that, for any a, b, the set $S \cap \{(u,v) \in \mathbb{R}^{d+m} : a \le u \le b\}$ is connected. Let $f \in C(S)$. Then, for any $(u_1, v_1) \in S$, $(u_2, v_2) \in S$, and $z \in [f(u_1,v_1), f(u_2,v_2)]$, there exists $(u,v) \in S$ such that $u \in [u_1, u_2]$ and $f(u,v) = z$.*

`Proof`. The statement immediately follows from the connectedness of $S \cap \{(u,v) \in \mathbb{R}^{d+m} : u_1 \le u \le u_2\}$. □

To conclude this subsection, we find the explicit form of A- and B-operators in a particular case. Let $\mu = \delta_\lambda$, where $\lambda \in (0,1)$, i.e. we deal with Tail V@R. To indicate this, below we replace μ by λ in the notation. Let $D_0 = D_1 = (0, \infty)$ and $P(x) = \mathsf{Law}\, x\xi$, where ξ is a strictly positive integrable random variable such that $\mathsf{Law}\, \xi$ is continuous, $\operatorname{supp} \mathsf{Law}\, \xi = \mathbb{R}_+$, and $u_\lambda(\xi) < 1 < -u_\lambda(-\xi)$.

Clearly, there exists a unique pair of numbers $0 < a < b < \infty$ such that $\mathsf{P}(\xi \in (a,b)) = \lambda$ and $\mathsf{E}(\xi \,|\, \xi \in (a,b)) = 1$; there exists a unique pair of numbers $0 < c < d < \infty$ such that $\mathsf{P}(\xi \notin (c,d)) = \lambda$ and $\mathsf{E}(\xi \,|\, \xi \notin (c,d)) = 1$.

Lemma 4.5. **(i)** *If $f \in \mathcal{L}(0,\infty)$ is convex, then*

$$A_{\lambda,P} f(x) = \int_{\mathbb{R}_+} f(xy) Q(dy), \quad x \in (0,\infty), \tag{4.13}$$

$$B_{\lambda,P} f(x) = \left\{ -\frac{f(bx) - f(ax)}{(b-a)x} \right\}, \quad x \in (0,\infty), \tag{4.14}$$

where $Q = \mathsf{Law}(\xi \,|\, \xi \in (a,b))$.

(ii) *If $f \in \mathcal{L}(0,\infty)$ is concave, then $A_{\lambda,P} f$ and $B_{\lambda,P} f$ have the same form with a, b replaced by c, d and the sign "$\in$" replaced by "$\notin$".*

`Proof`. We will prove only (i). First, we verify (4.13). Without loss of generality, $x = 1$. Let us first assume additionally that f is strictly convex. As shown in the proof of Theorem 4.2 (iii), we can find $Z_* \in \mathcal{D}_{\lambda,P}$ such that $\mathsf{E}_P Z_* X = 1$ and $\mathsf{E}_P Z_* f(X) = A_{\lambda,P} f(1)$, where $P = \mathsf{Law}\, \xi$. Let us prove that $Z_* = \lambda^{-1} I(X \in (a,b))$ P-a.s. Assume the contrary. We can write $Z_* = \varphi(X)$. Then there exist $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ such that

$$\begin{aligned} &P(\{\varphi > 0\} \cap (\alpha_1, \alpha_2)) > 0, \\ &P(\{\varphi < \lambda^{-1}\} \cap (\alpha_2, \alpha_3)) > 0, \\ &P(\{\varphi > 0\} \cap (\alpha_3, \alpha_4)) > 0. \end{aligned}$$

For $h_1, h_2, h_3 \in [0, \lambda^{-1}]$, let us set

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x), & x \notin (\alpha_1, \alpha_4], \\ \varphi(x) \wedge h_1, & x \in (\alpha_1, \alpha_2], \\ \varphi(x) \vee h_2, & x \in (\alpha_2, \alpha_3], \\ \varphi(x) \wedge h_3, & x \in (\alpha_3, \alpha_4]. \end{cases}$$

We can find h_1, h_2, h_3 such that

$$\begin{aligned} &P(\{\widetilde{\varphi} < \varphi\} \cap (\alpha_1, \alpha_2)) > 0, \\ &P(\{\widetilde{\varphi} > \varphi\} \cap (\alpha_2, \alpha_3)) > 0, \\ &P(\{\widetilde{\varphi} < \varphi\} \cap (\alpha_3, \alpha_4)) > 0, \\ &\mathsf{E}_P \widetilde{\varphi}(X) = \mathsf{E}_P \varphi(X) = 1, \\ &\mathsf{E}_P X \widetilde{\varphi}(X) = \mathsf{E}_P X \varphi(X) = 1. \end{aligned}$$

Consider the affine function $\widetilde{f}$ that coincides with f at α_2 and α_3. The above equalities imply that $\mathsf{E}_P(\widetilde{\varphi}(X) - \widetilde{\varphi}(X))\widetilde{f}(X) = 0$. Furthermore, as f is strictly convex, $\widetilde{f} < f$ on (α_1, α_2), $\widetilde{f} > f$ on (α_2, α_3), and $\widetilde{f} > f$ on (α_3, α_4). Consequently, $\mathsf{E}_P(\widetilde{\varphi}(X) - \varphi(X))f(X) < 0$. Thus, we have found $\widetilde{Z} = \widetilde{\varphi}(X) \in \mathcal{D}_{\lambda,P}$ such that $\mathsf{E}_P\widetilde{Z}X = 1$ and $\mathsf{E}_P\widetilde{Z}f(X) < \mathsf{E}_P Z_* f(X)$. But on the other hand, it follows from (4.9) that

$$A_{\lambda,P}f(1) = \inf_{Z \in \mathcal{D} \cap \mathcal{M}_{\lambda,P}} \mathsf{E}Zf(X), \qquad (4.15)$$

where $\mathcal{M} = \{Z : \mathsf{E}ZX = 1\}$. The obtained contradiction shows that $Z_* = \lambda^{-1}I(X \in (a,b))$ P-a.s., which yields (4.13) for a strictly convex f.

Let us now prove (4.13) in the general case. Take $Z_* = \lambda^{-1}I(X \in (a,b))$. Find a strictly convex function $\widetilde{f}$ of linear growth. Then the function $f_\varepsilon = f + \varepsilon\widetilde{f}$ is strictly convex and, in view of (4.15), the result proved above shows that $\mathsf{E}_P Z f_\varepsilon(X) \ge \mathsf{E}Z_* f_\varepsilon(X)$ for any $Z \in \mathcal{D} \cap \mathcal{M}_{\lambda,P}$. Passing on to the limit as $\varepsilon \downarrow 0$, we get $\mathsf{E}_P Z f(X) \ge \mathsf{E}_P Z_* f(X)$ for any $Z \in \mathcal{D} \cap \mathcal{M}_{\lambda,P}$. Employing (4.15) once more, we get (4.13).

Let us now prove (4.14). Without loss of generality, $x = 1$. Consider the function

$$g(y) = \inf\{z : (y,z) \in G(1)\} = \inf\{\mathsf{E}Zf(X) : Z \in \mathcal{D}_{\lambda,P},\ \mathsf{E}_P ZX = y + 1\},$$

where $G(x)$ is defined by (4.7). Repeating the proof of (4.13), we check that $g = g_1 \circ g_2^{-1}$, where

$$g_1(y) = \lambda^{-1}\int_{q_y}^{q_{y+\lambda}} f(x)P(dx), \quad y \in (0, 1-\lambda),$$
$$g_2(y) = \lambda^{-1}\int_{q_y}^{q_{y+\lambda}} xP(dx), \quad y \in (0, 1-\lambda),$$

and q_y is the y-quantile of P. Due to (4.10),

$$B_{\lambda,P}f(1) = \beta(G(1)) = \{-g'(0)\} = \left\{-\frac{f(b) - f(a)}{b-a}\right\}.$$

4.2 Pricing and Hedging

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P})$ be a filtered probability space. Let

$$\begin{aligned}\mathcal{D}_n = \{Z \in L^0(\mathcal{F}_n) : Z \ge 0,\ \mathsf{E}(Z|\mathcal{F}_{n-1}) = \mu((0,1]),\\ \text{and } \mathsf{E}((Z-x)^+|\mathcal{F}_{n-1}) \le \Phi_\mu(x)\ \forall x \in \mathbb{R}_+\}, \quad n = 1,\dots,N,\end{aligned}$$

where μ is a positive measure on $(0,1]$ with $\mu((0,1]) \le 1$ and Φ_μ is given by (4.4). Thus, the risk measure we consider is the *dynamic Weighted V@R* (see [10]).[8] Let $(S_n)_{n=0,\dots,N}$ be a d-dimensional adapted process. We will assume that there exists $m \in \mathbb{Z}_+$, an m-dimensional adapted process $(\Theta_n)_{n=0,\dots,N}$, and a collection of measurable sets $(D_n)_{n=0,\dots,N} \subseteq \mathbb{R}^{d+m}$ such that $(S_n, \Theta_n) \in D_n$ a.s. for any n. Furthermore, we assume that, for each $n = 1,\dots,N$, there exists a family of probability measures $(P_n(x,y))_{(x,y)\in D_{n-1}}$ on D_n satisfying conditions (4.5), (4.6) with D_0, D_1 replaced by D_{n-1}, D_n and such that

$$\mathsf{Law}(S_n, \Theta_n | \mathcal{F}_{n-1}) = P_n(S_{n-1}, \Theta_{n-1}) \quad \text{a.s.}$$

[8] In view of (4.2)–(4.4), this is a straightforward dynamic extension of the static Weighted V@R.

We will consider a stream of cash flows of the form $F_n = f_n(S_n, \Theta_n)$, $n = 0, \dots, N$, where $f_n \in \mathcal{L}(D_n)$.

Define the functions $(g_n)_{n=0,\dots,N}$ by: $g_N = 0$, $g_{n-1} = A_{\mu,P_n}(f_n + g_n)$ (due to Theorem 4.2 (i), these functions are correctly defined).

Theorem 4.6. **(i)** *We have*

$$\begin{aligned}&\underline{V}_n(F) = g_n(S_n, \Theta_n), \quad n = 0, \dots, N,\\&\underline{\mathcal{H}}_n(F) = L^0(\mathcal{F}_{n-1}, B_{\mu,P_n}(f_n + g_n)(S_{n-1}, \Theta_{n-1})), \quad n = 1, \dots, N.\end{aligned}$$

(ii) *If* $\mu((0,1)) > 0$, $\operatorname{supp} P_n(x,y) \cap D_n$ *is strongly connected for any* n, x, y, *and* $\mathsf{Law}_{P_n(x,y)}\langle h, X\rangle$ *is continuous for any* n, x, y, *and* $h \in \mathbb{R}^d \setminus \{0\}$, *then each* $\underline{\mathcal{H}}_n(F)$ *is a singleton.*

`Proof`. **(i)** Denote $G_N(x,y) = 0$,

$$\begin{aligned}G_{n-1}(x,y) = \operatorname{cl}\{&\mathsf{E}_{P_n(x,y)} Z(X - x, (f_n + g_n)(X, Y)) :\\&Z \in \mathcal{D}_{\mu,P_n(x,y)}\}, \quad (x,y) \in D_{n-1},\ n = 1, \dots, N.\end{aligned}$$

Let us prove, going backwards from N to 0, that $\underline{G}_n(F) = G_n(S_n, \Theta_n)$ and $\underline{a}_n(F) = g_n(S_n, \Theta_n)$, where $\underline{G}_n(F)$ and $\underline{a}_n(F)$ are defined by (3.3) and (3.4). Suppose that these equalities are true for n and let us prove them for $n-1$. As shown in the proof of Theorem 4.2, the map $D_{n-1} \ni (x,y) \mapsto G_{n-1}(x,y)$ is continuous, and therefore, $G_{n-1}(S_{n-1}, \Theta_{n-1})$ is $\mathcal{F}_{n-1}$-measurable. It is known that $u_\mu(X)$ depends only on the distribution of X, so that there exists a functional $\widetilde{u}_\mu$ defined on distributions such that $u_\mu(X) = \widetilde{u}_\mu(\mathsf{Law}\, X)$, $X \in L^0$. We have

$$\begin{aligned}\varphi_{(h_1,h_2)}(\underline{G}_{n-1}(F)) &= \operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(\langle h_1, \Delta S_n\rangle + h_2(f_n + g_n)(S_n, \Theta_n)) \,|\, \mathcal{F}_{n-1})\\&= \widetilde{u}_\mu(\mathsf{Law}_{P_n(x,y)}(\langle h_1, X - x\rangle + h_2(f_n + g_n)(X, Y)))_{(x,y)=(S_{n-1},\Theta_{n-1})}\\&= u_{\mu,P_n(x,y)}(\langle h_1, X - x\rangle + h_2(f_n + g_n)(X, Y))\\&= \varphi_{(h_1,h_2)}(G_{n-1}(S_{n-1}, \Theta_{n-1})), \quad h_1 \in \mathbb{R}^d,\ h_2 \in \mathbb{R},\end{aligned}$$

where φ is the support function given by (a.2). The first equality here follows from Lemma A.4 and the second one follows from [10; Lem. 2.5]. Consequently, $\underline{G}_{n-1}(F) = G_{n-1}(S_{n-1}, \Theta_{n-1})$ a.s. The equality $\underline{a}_{n-1}(F) = g_{n-1}(S_{n-1}, \Theta_{n-1})$ follows from (4.9).

Now, the desired result follows from Theorem 3.2 combined with (4.9) and (4.10).

(ii) This statement is a consequence of (i) and Theorem 4.2. □

`Remark`. The vector of sensitivities at time n of the portfolio to the underlying assets might be defined as $\Delta_n = \left(\frac{\partial g_n(S_n,\Theta_n)}{\partial S_n^i}\right)_{i=1,\dots,d}$. Suppose that $\underline{\mathcal{H}}_n$ consists of a unique element H_n. Then it is not true that $H_n = -\Delta_n$. This is in contrast with the continuous-time situation, where this equality is true in some natural models (see [21]). However, under some conditions, a discrete-time model should "converge" to a continuous-time limit, and then the above equality should approximately be true. □

Example 4.7 (GARCH). Let μ be such that $\mu((0,1)) > 0$ and $u_\mu(-\xi) > -\infty$ for any lognormal random variable ξ (as an example, $\mu = \delta_\lambda$ with $\lambda \in (0,1)$ satisfies this condition). Let S be a d-dimensional adapted strictly positive componentwise process and Θ be a process with values in the set of symmetric positively definite $d\times d$-matrices such that

$$\mathsf{Law}(\ln S_n\,|\,\mathcal{F}_{n-1}) = \mathcal{N}(\ln S_{n-1}-\overline{\Theta}_{n-1}/2,\Theta_{n-1}),\quad n=1,\dots,N,$$
$$\Theta_n^{ij} = \alpha\Delta\ln S_n^i\Delta\ln S_n^j+\beta\Theta_{n-1}^{ij}+\Gamma^{ij},\quad i,j=1,\dots,d,\ n=1,\dots,N,$$

where $\mathcal{N}$ is the Gaussian distribution, $\ln S_n$ is taken componentwise, $\overline{\Theta}_{n-1}^i = \overline{\Theta}_{n-1}^{ii}$ $\alpha,\beta>0$, $\alpha+\beta<1$, and Γ is a symmetric strictly positively definite matrix (we subtract the term $\overline{\Theta}_{n-1}/2$ from $\ln S_{n-1}$ in order for the process S to be a martingale; this, in turn, is needed for the NGD condition).

To embed this model in the framework of this subsection, take $D_n=(0,\infty)^d\times M$, where $M\subset\mathbb{R}^{d\times d}$ is the set of symmetric strictly positively definite $d\times d$-matrices (note that any Θ_n has this property), and $P_n(x,y)=\mathsf{Law}(e^\xi,\eta)$, where $\mathsf{Law}\,\xi=\mathcal{N}(\ln x-\overline{y}/2,y)$, $\eta^{ij}=\alpha\xi^i\xi^j+\beta y^{ij}+\Gamma^{ij}$, and e^ξ is taken componentwise. Let us check that all the assumptions of this subsection are satisfied.

Fix n, $(x,y)\in D_n$, and $\varepsilon>0$. There exists a lognormal (possibly, shifted) random variable ξ such that $\mathsf{Law}\,\xi$ stochastically dominates $\mathsf{Law}_{P_n(x',y')}|X|$ for any $(x',y')\in B_\varepsilon(x,y)\cap D_{n-1}$. Then $\mathsf{Law}\,\xi I(\xi>k)$ stochastically dominates $\mathsf{Law}_{P_n(x',y')}|X|I(|X|>k)$, so that

$$0\ge u_{\mu,P(x',y')}(-|X|I(|X|>k))\ge u_\mu(-\xi I(\xi>k))\xrightarrow[k\to\infty]{}0.$$

The convergence here follows from [7; Prop. 2.6]. Thus, we have checked (4.5).

Condition (4.6) follows from the line

$$u_{\mu,P_n(x,y)}(\langle h,X-x\rangle)<\mathsf{E}_{P_n(x,y)}\langle h,X-x\rangle=0,\ \ h\in\mathbb{R}^d\setminus\{0\},\ (x,y)\in D_{n-1},\ n=1,...,N.$$

The inequality here follows from the condition $\mu((0,1))>0$ and the non-degeneracy of y.

Finally, the conditions of part (ii) of the above theorem are obviously satisfied. In particular, $\underline{\mathcal{H}}_n(F)$ is a singleton for any F of the form described above. □

`Remark`. The model of the above example is GARCH(1,1). GARCH(p,q) model can also be embedded in the framework of this subsection simply by extending the process Θ. □

4.3 Convex and Concave Payoffs

Let $(\Omega,\mathcal{F},(\mathcal{F}_n)_{n=0,\dots,N},\mathsf{P})$ be a filtered probability space and $\mathcal{D}_n$ be the same as in the previous subsection with $\mu=\delta_\lambda$, where $\lambda\in(0,1)$, i.e. the risk measure we consider is the *dynamic Tail V@R*. Let $S_n=S_0e^{\eta_1+\dots+\eta_n}$, where S_0 is strictly positive and η_n is independent of $\mathcal{F}_{n-1}$ for any n. We assume that each η_n has a continuous distribution with $\operatorname{supp}\mathsf{Law}\,\eta_n=\mathbb{R}$ and $u_\lambda(e^{\eta_n})<1<-u_\lambda(-e^{\eta_n})$. Let $F_n=f_n(S_n)$, where $f_n\in\mathcal{L}(0,\infty)$.

Clearly, for each n, there exists a unique pair of numbers $0<a_n<b_n<\infty$ such that $\mathsf{P}(e^{\eta_n}\in(a_n,b_n))=\lambda$ and $\mathsf{E}(e^{\eta_n}\,|\,e^{\eta_n}\in(a_n,b_n))=1$; for each n, there exists a unique pair of numbers $0<c_n<d_n<\infty$ such that $\mathsf{P}(e^{\eta_n}\notin(c_n,d_n))=\lambda$ and $\mathsf{E}(e^{\eta_n}\,|\,e^{\eta_n}\notin(c_n,d_n))=1$. Let $\eta_1',\dots,\eta_N'$ be independent random variables with $\mathsf{Law}\,\eta_n'=\mathsf{Law}(\eta_n\,|\,\eta_n\in(\ln a_n,\ln b_n))$; let $\eta_1'',\dots,\eta_N''$ be independent random variables with $\mathsf{Law}\,\eta_n''=\mathsf{Law}(\eta_n\,|\,\eta_n\notin(\ln c_n,\ln d_n))$.

Theorem 4.8. **(i)** *Suppose that each* f_n *is convex.*[9] *Then* $\underline{V}_n(F) = g_n(S_n)$, *where* $g_n(x) = \widetilde{g}_n(\ln x)$,

$$\widetilde{g}_n(x) = \sum_{m=n+1}^{N} \mathsf{E}\widetilde{f}_n(x + \eta'_{n+1} + \dots + \eta'_m), \quad x \in \mathbb{R},\ n = 0, \dots, N,$$

and $\widetilde{f}_n(x) = f_n(e^x)$. *The set* $\underline{\mathcal{H}}_n(F)$ *consists of a unique element*

$$\underline{H}_n = -\frac{(f_n + g_n)(b_n S_{n-1}) - (f_n + g_n)(a_n S_{n-1})}{(b_n - a_n)S_{n-1}}, \quad n = 1, \dots, N.$$

If moreover f_{n_0} *is strictly convex for some* n_0, *then*[10]

$$\mathcal{X}_n(F; \mathcal{D} \cap \mathcal{M}) = \{\lambda^{-1} I(\eta_n \in (\ln a_n, \ln b_n))\}, \quad n = 1, \dots, n_0.$$

(ii) *Suppose that each* f_n *is concave. Then* $\underline{V}_n(F)$, $\underline{\mathcal{H}}_n(F)$, *and* $\mathcal{X}_n(F; \mathcal{D} \cap \mathcal{M})$ *have the same form with* η'_n, a_n, b_n *replaced by* η''_n, c_n, d_n *and the sign "*$\in$*" replaced by "*$\notin$*".*

`Proof.` We will prove only (i). The representations for $\underline{V}_n(F)$ and $\underline{\mathcal{H}}_n(F)$ follow from Lemma 4.5 combined with Theorem 4.6. Let us prove the representation for $\mathcal{X}_n(F; \mathcal{D} \cap \mathcal{M})$. Fix $n \le n_0$. Applying Theorems 3.2 and 4.6, we can write

$$\begin{aligned}
&\operatorname*{essinf}_{Z \in \mathcal{D}_n \cap \mathcal{M}_n} \mathsf{E}(Z(f_n + g_n)(S_n) \,|\, \mathcal{F}_{n-1}) \\
&= \operatorname*{essinf}_{Z \in \mathcal{D}_n \cap \mathcal{M}_n} \mathsf{E}(Z(F_n + u_n(F; \mathcal{D} \cap \mathcal{M})) \,|\, \mathcal{F}_{n-1}) \\
&= u_{n-1}(F; \mathcal{D} \cap \mathcal{M}) = \underline{V}_{n-1}(F) = u_{n-1}(\langle \underline{H}, \Delta S \rangle + F) \\
&= \operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(\underline{H}_n \Delta S_n + F_n + u_n(\langle \underline{H}, \Delta S \rangle + F)) \,|\, \mathcal{F}_{n-1}) \\
&= \operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(\underline{H}_n \Delta S_n + (f_n + g_n)(S_n)) \,|\, \mathcal{F}_{n-1}),
\end{aligned}$$

where $\underline{H} = (\underline{H}_n)_{n=1,\dots,N}$. Consequently,

$$\begin{aligned}
\mathcal{X}_n(F; \mathcal{D} \cap \mathcal{M}) &= \operatorname*{argessmin}_{Z \in \mathcal{D}_n \cap \mathcal{M}_n} \mathsf{E}(Z(f_n + g_n)(S_n) \,|\, \mathcal{F}_{n-1}) \\
&\subseteq \operatorname*{argessmin}_{Z \in \mathcal{D}_n} \mathsf{E}(Z(\underline{H}_n \Delta S_n + (f_n + g_n)(S_n)) \,|\, \mathcal{F}_{n-1}) \\
&= \operatorname*{argessmin}_{Z \in \mathcal{D}_n} \mathsf{E}(Z\psi(S_{n-1}, \xi_n) \,|\, \mathcal{F}_{n-1}),
\end{aligned}$$

where $\xi_n = e^{\eta_n}$ and

$$\psi(x, y) = (f_n + g_n)(xy) - \frac{(f_n + g_n)(b_n x) - (f_n + g_n)(a_n x)}{b_n - a_n}(y - 1), \quad x, y \in (0, \infty).$$

Going backwards from n_0 to n, we check that g_n is strictly convex. Thus, ψ is strictly convex in y. Obviously, $\psi(x, a_n) = \psi(x, b_n)$ for any x. Due to the strict convexity,

[9] In particular, this assumption is satisfied if F corresponds to a call option, in which case $f_n = 0$ for $n = 0, \dots, N-1$ and $f_N(x) = (x - K)^+$.

[10] Recall that, according to the results of Section 3, the sets $\mathcal{X}_n(F; \mathcal{D} \cap \mathcal{M})$ are needed to calculate price contributions, market-adjusted risk contributions, and market-adjusted capital allocations.

$\psi(x,y)<\psi(x,a_n)$ for $y\in(a_n,b_n)$ and $\psi(x,y)>\psi(x,a_n)$ for $y\notin[a_n,b_n]$. Now, it follows from [10; Lem. 4.5] that

$$\operatorname*{argessmin}_{Z\in\mathcal{D}_n}\mathsf{E}(Z\psi(S_{n-1},\xi_n)\,|\,\mathcal{F}_{n-1})=\{\lambda^{-1}I(\xi_n\in(a_n,b_n))\}=\{\lambda^{-1}I(\eta_n\in(\ln a_n,\ln b_n))\}.$$

As $\mathcal{X}_n(F;\mathcal{D}\cap\mathcal{M})$ is non-empty, we get the desired representation. □

`Remarks`. (i) As seen from the above proof, the inclusion

$$\{\lambda^{-1}I(\eta_n\in(\ln a_n,\ln b_n))\}\subseteq\mathcal{X}_n(F;\mathcal{D}\cap\mathcal{M}),\quad n=1,\dots,N$$

employs only the convexity of f_n, while the strict convexity is needed only for the reverse inclusion. A similar remark applies to concave functions.

(ii) If each f_n is strictly convex, then the measure Q appearing in formula (3.8) for the price contribution gets a very simple form: $\mathsf{Q}=\mathsf{P}(\,\cdot\,|\,\eta_n\in(\ln a_n,\ln b_n)\;\forall n)$. A similar remark applies to concave functions.

(iii) The results of this subsection can be extended to $\mu=\alpha\delta_\lambda+(1-\alpha)\delta_1$ (this risk measure is important for passing to the continuous-time limit). One should redefine a_n,b_n using the equalities $\mathsf{P}(e^{\eta_n}\in(a_n,b_n))=\lambda$, $\alpha\mathsf{E}(e^{\eta_n}\,|\,e^{\eta_n}\in(a_n,b_n))+(1-\alpha)\mathsf{E}e^{\eta_n}=1$ and redefine η_n' by $\mathsf{Law}\,\eta_n'=\alpha\,\mathsf{Law}(\eta_n\,|\,\eta_n\in(\ln a_n,\ln b_n))+(1-\alpha)\,\mathsf{Law}\,\eta_n$. Then, for convex f_n, the representations for $\underline{V}_n(F)$ and $\underline{\mathcal{H}}_n(F)$ remain the same, while the representation for $\mathcal{X}_n(F;\mathcal{D}\cap\mathcal{M})$ gets the form

$$\mathcal{X}_n(F;\mathcal{D}\cap\mathcal{M})=\{\alpha\lambda^{-1}I(\eta_n\in(\ln a_n,\ln b_n))+1-\alpha\},\quad n=1,\dots,n_0.$$

A similar remark applies to concave functions. □

4.4 Numerical Algorithm

Consider the model of Subsection 4.2 and assume that each D_n is finite and $\mathcal{F}_n=\sigma(S_k,\Theta_k:k\le n)$. Define the real-valued functions $(g_n)_{n=0,\dots,N}$ and the set-valued functions $(H_n)_{n=0,\dots,N-1}$, $(X_n)_{n=0,\dots,N-1}$ by: $g_N=0$,

$$\begin{aligned}
g_{n-1}(a)&=\max_{h\in\mathbb{R}^d,\,q\in\mathbb{R}}\Big[q-\lambda^{-1}\sum_{b\in D_n}(q-\psi_n(a,b,h))^+p_n(a,b)\Big],\quad a\in D_{n-1},\\
H_{n-1}(a)&=\{h\in\mathbb{R}^d:\exists q\in\mathbb{R}:(h,q)\in M_{n-1}(a)\},\quad a\in D_{n-1},\\
X_{n-1}(a)&=\Big\{(z(b))_{b\in D_n}:0\le z(b)\le\lambda^{-1},\ \sum_{b\in D_n}z(b)p_n(a,b)=1,\\
&\qquad z(b)=0\text{ if }\psi_n(a,b,h)>q,\ z(b)=\lambda^{-1}\text{ if }\psi_n(a,b,h)<q,\\
&\qquad\text{and }\mathrm{pr}_{\mathbb{R}^d}\sum_{b\in D_n}(b-a)z(b)p_n(a,b)=0\Big\},\quad a\in D_{n-1},
\end{aligned}$$

where $p_n(a,b)=P_n(a)(\{b\})$,

$$\begin{aligned}
\psi_n(a,b,h)&=\langle h,\mathrm{pr}_{\mathbb{R}^d}(b-a)\rangle+f_n(b)+g_n(b),\quad a\in D_{n-1},\ b\in D_n,\ h\in\mathbb{R}^d,\\
M_{n-1}(a)&=\operatorname*{argmax}_{h\in\mathbb{R}^d,\,q\in\mathbb{R}}\Big[q-\lambda^{-1}\sum_{b\in D_n}(q-\psi_n(a,b,h))^+p_n(a,b)\Big],\quad a\in D_{n-1},
\end{aligned}$$

and in the definition of $X_{n-1}(a)$, (h,q) is an arbitrary element of $M_{n-1}(a)$ (due to Lemma 4.10, $M_{n-1}(a)\neq\emptyset$ and $X_{n-1}(a)$ does not depend on the choice of (h,q)). Let us remark that typically the first four conditions in the definition of $X_{n-1}(a)$ already define it uniquely, and in this case (due to non-emptiness of $X_{n-1}(a)$) the condition $\mathrm{pr}_{\mathbb{R}^d}\sum(b-a)z(b)p_n(a,b)=0$ can be dropped.

In the statement below, we use the convention: an $\mathcal{F}_n$-measurable random variable Z is written as a function of n arguments: $Z(a_0,\dots,a_n)=Z|_{\{(S_0,\Theta_0)=a_0,\dots,(S_n,\Theta_n)=a_n\}}$. We use the notation $A_n(a)=\{(S_n,\Theta_n)=a\}$.

Proposition 4.9. *We have*

$$
\begin{aligned}
\underline{V}_n(F) &= g_n(a) \text{ on } A_n(a), \quad a\in D_n,\ n=0,\dots,N,\\
\underline{\mathcal{H}}_n(F) &= \{H\in L^0(\mathcal{F}_{n-1})\colon H_n\in H_{n-1}(a) \text{ on } A_{n-1}(a)\ \forall a\in D_{n-1}\},\quad n=1,\dots,N,\\
\mathcal{X}_n(F;\mathcal{M}\cap\mathcal{D}\} &= \{Z\in L^0(\mathcal{F}_n)\colon Z(a_0,\dots,a_{n-1},\cdot)\in X_{n-1}(a_{n-1})\ \forall a_k\in D_k\},\quad n=1,\dots,N.
\end{aligned}
$$

Proof. This statement follows from Theorem 4.6 combined with Lemma 4.10. □

The lemma below goes back to Pflug [23], Rockafellar and Uryasev [24].

Lemma 4.10. *Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space, $\lambda\in(0,1)$, η be an integrable random variable, and ξ be a d-dimensional integrable random vector such that $u_\lambda(\langle h,\xi\rangle)<0$ for any $h\in\mathbb{R}^d\setminus\{0\}$. Denote $\Psi(h)=\langle h,\xi\rangle+\eta$, $h\in\mathbb{R}^d$. Then*

$$\sup_{h\in\mathbb{R}^d} u_\lambda(\Psi(h)) = \sup_{h\in\mathbb{R}^d,\,q\in\mathbb{R}} (q-\lambda^{-1}\mathsf{E}(q-\Psi(h))^+)<\infty, \tag{4.16}$$

$$\operatorname*{argmax}_{h\in\mathbb{R}^d} u_\lambda(\Psi(h)) = \big\{h\in\mathbb{R}^d : \exists q\in\mathbb{R}\colon (h,q)\in \operatorname*{argmax}_{h\in\mathbb{R}^d,\,q\in\mathbb{R}}(q-\lambda^{-1}\mathsf{E}(q-\Psi(h))^+)\big\}\neq\emptyset. \tag{4.17}$$

Furthermore, for any $(h,q)\in\mathrm{argmax}_{h,q}(q-\lambda^{-1}\mathsf{E}(q-\Psi(h))^+)$, we have

$$
\begin{aligned}
\operatorname*{argmin}_{Z\in\mathcal{D}_\lambda,\,\mathsf{E}Z\xi=0} \mathsf{E}Z\eta = \{Z\in\mathcal{D}_\lambda : Z=0 \text{ a.e. on } \{\Psi(h)>q\},&\\
Z=\lambda^{-1} \text{ a.e. on } \{\Psi(h)<q\}, \text{ and } \mathsf{E}Z\xi=0\}\neq\emptyset.&
\end{aligned} \tag{4.18}
$$

Proof. Fix h and denote $f(q)=q-\lambda^{-1}\mathsf{E}(q-\Psi(h))^+$. It follows from the equalities $f'_-(q)=1-\lambda^{-1}\mathsf{P}(\Psi(h)<q)$, $f'_+(q)=1-\lambda^{-1}\mathsf{P}(\Psi(h)\le q)$ that the maximum of $f(q)$ is attained exactly at the set of λ-quantiles of $\Psi(h)$. For any q from this set, we have

$$q-\lambda^{-1}\mathsf{E}(q-\Psi(h))^+ = q-\lambda^{-1}\mathsf{P}(\Psi(h)<q)+\lambda^{-1}\mathsf{E}\Psi(h)I(\Psi(h)<q)=u_\lambda(\Psi(h)).$$

In the last equality, we used a well-known representation of u_λ; see, for example, [9; Prop. 2.7]. This proves the equalities in (4.16), (4.17).

It follows from the relation

$$\sup_{|h|=n} n^{-1}u_\lambda(\Psi(h)) = \sup_{|h|=1} u_\lambda(\langle h,\xi\rangle+n^{-1}\eta) \xrightarrow[n\to\infty]{} \sup_{|h|=1} u_\lambda(\langle h,\xi\rangle)<0$$

that $\sup_{|h|\ge n} u_\lambda(\Psi(h)) \xrightarrow[n\to\infty]{} -\infty$. As $u_\lambda(\Psi(h))$ is continuous in h, it attains its maximum.

Let us now check (4.18). In view of Lemma 4.3, its right-hand side is $\mathrm{argmin}_{Z\in\mathcal{D}_\lambda}\mathsf{E}Z\Psi(h)\cap\{Z : \mathsf{E}Z\xi=0\}$. Now, the equality in (4.18) follows from the line

$$u_\lambda(\Psi(h)) = \max_{h'\in\mathbb{R}^d} u_\lambda(\langle h',\xi\rangle+\eta) = \inf_{Z\in\mathcal{D}_\lambda,\,\mathsf{E}Z\xi=0} \mathsf{E}Z\eta,$$

which is, in turn, a consequence of Theorem 3.2. The non-emptiness of the left-hand side of (4.18) follows from a compactness argument. □

Remark. One can provide a discretized version of the results of Subsection 4.2 directly, without using the Pflug-Rockafellar-Uryasev method, i.e. instead of the problem

$$q - \lambda^{-1} \sum_{b \in D_n} (q - \psi_n(a,b,h))^+ p_n(a,b) \xrightarrow[h \in \mathbb{R}^d,\, q \in \mathbb{R}]{} \max, \tag{4.19}$$

one can consider at each point $a \in D_{n-1}$ the problem $u_{\mu,P_n(a)}(\Psi_n(a,h)) \xrightarrow[h \in \mathbb{R}^d]{} \max$, where $\Psi_n(a,h)$ is a random variable taking the values $\psi_n(a,b,h)$ with probabilities $p_n(a,b)$. This would allow one to consider not only the dynamic Tail V@R, but the general dynamic Weighted V@R. However, the use of the Pflug-Rockafellar-Uryasev method has the following advantages: it leads to simpler formulas; it provides explicitly the level q needed to define $X_{n-1}(a)$; it leads to a convex optimization problem (4.19), to which standard numerical methods can be applied. □

5 Conclusion

Let us discuss some issues of the practical application of the proposed technique. Suppose that a company has a large portfolio of European options, each option depending only on the value of an underlying asset at some fixed date. Then, within the Markov model (this class of models includes virtually all the natural ones), its risk-based price process is given by $g_n(S_n, \Theta_n)$, where the functions g_n admit closed-form expressions in some particular cases and a numerical calculation procedure in the general case. The number of computations depends linearly on the number of contracts in the portfolio. Furthermore, if the basic risk measure is the dynamic Tail V@R, then this procedure can be simplified and accelerated by employing the Pflug-Rockafellar-Uryasev method.

If a new small subportfolio is added to the portfolio, then the change in the price is well approximated by the price contribution. In typical situations, it is given simply by the expectation of the cumulative cash flow of the subportfolio with respect to the extreme measure of the portfolio (a numeric procedure for calculating this measure has also been provided). This allows one not to repeat the whole procedure for calculating the functions g_n each time the portfolio is rebalanced; this procedure can be performed only periodically when the portfolio has essentially been changed. Furthermore, the technique of price contributions allows one to price also portfolios including path-dependent options provided that the part of these options is small: the price contributions of these options can be estimated using the Monte Carlo method.

Suppose now that, apart from the portfolio of options, the company has a portfolio of primary assets, which are perfectly liquid. Then the market-adjusted risk of the joint portfolio is calculated by the same backward procedure as the one used for the price, with the only difference at the last step (passing on from time 1 to time 0). When calculating the risk at time 1, one should not repeat the whole procedure (provided that the illiquid part of the portfolio remains unchanged); only passing from time 2 to time 1 should be done anew. All the remarks concerning price contributions are carried over to risk contributions.

The described method of risk measurement differs from the classical one, in which the sensitivity coefficients are first calculated and then risk is estimated. Here, the same procedure (with the only difference at the last step) applies both to pricing and to risk measurement.

Appendix

Let $\mathcal{C}$ be the set of non-empty convex compacts in $\mathbb{R}^{d+1}$ endowed with the Hausdorff metrics

$$\rho(C_1, C_2) = \sup_{x_n \in C_n} \|x_1 - x_2\| \tag{a.1}$$

and the corresponding Borel σ-field. Below φ_h denotes the support function

$$\varphi_h : \mathcal{C} \ni C \longmapsto \min_{x \in C} \langle h, x \rangle, \quad h \in \mathbb{R}^{d+1} \tag{a.2}$$

and α, β are the maps defined in (3.1), (3.2).

Lemma A.1. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $C : \Omega \to \mathcal{C}$ be a measurable map such that $0 \in \mathrm{pr}_{\mathbb{R}^d} C$ a.s. Then, for any $\varepsilon > 0$, there exists a measurable map $h : \Omega \to \mathbb{R}^d$ such that $\varphi_{(h,1)}(C) \geq \alpha(C) - \varepsilon$ a.s.*

Proof. First, we prove the statement for a non-random C such that 0 belongs to the interior of $\mathrm{pr}_{\mathbb{R}^d} C$. By the Hahn-Banach theorem, there exists $\widetilde{h} \in \mathbb{R}^{d+1}$ such that $\langle \widetilde{h}, (0, \alpha(C)) \rangle = \varphi_{\widetilde{h}}(C)$. Clearly, $\widetilde{h}^{d+1} > 0$, so that, without loss of generality, $\widetilde{h}^{d+1} = 1$. Taking $h = \mathrm{pr}_{\mathbb{R}^d} \widetilde{h}$ does the job.

For a general non-random C, denote by C_n the closed n^{-1}-neighborhood of C. Then $\alpha(C_n) \to \alpha(C)$, $\varphi_{(h,1)}(C_n) \leq \varphi_{(h,1)}(C)$, and the statement we need follows from the one proved above.

Consider now a random C. Due to [10; Lem. A.1], the function $\varphi_{(h,1)}(C(\omega))$ is measurable in the pair (h, ω). The map $\mathcal{C} \ni C \mapsto \alpha(C) \in (-\infty, \infty]$ is lower semicontinuous and therefore measurable. Hence, $\alpha(C(\omega)) - \varepsilon$ is measurable in ω. Thus,

$$A := \{(h, \omega) : \varphi_{(h,1)}(C(\omega)) \geq \alpha(C(\omega)) - \varepsilon\} \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{F}.$$

The measurable selection theorem, combined with the result for non-random C, yields the desired statement. □

Lemma A.2. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $C : \Omega \to \mathcal{C}$ be a measurable map such that $\beta(C) \neq \emptyset$ a.s. Then there exists a measurable map $h : \Omega \to \mathbb{R}^d$ such that $h \in \beta(C)$ a.s.*

The proof is similar to the proof of Lemma A.1.

Definition A.3. Let $(X_\lambda)_{\lambda \in \Lambda}$ be a family of $\mathbb{R}^{d+1}$-valued random vectors. The *essential closed convex hull* of $(X_\lambda)_{\lambda \in \Lambda}$ is a $\mathcal{C}$-valued random element C with the properties:

(a) for any λ, $X_\lambda \in C$ a.s.

(b) if C' is another random element with this property, then $C \subseteq C'$ a.s.

We use the notation $C = \mathrm{essconv}_{\lambda \in \Lambda} X_\lambda$.

Lemma A.4. *If $\mathrm{esssup}_\lambda \|X_\lambda\| < \infty$, then $\mathrm{essconv}_\lambda X_\lambda$ exists. Moreover, for any $h \in L^0(\mathbb{R}^{d+1})$,*

$$\varphi_h\big(\mathop{\mathrm{essconv}}_{\lambda \in \Lambda} X_\lambda\big) = \mathop{\mathrm{essinf}}_{\lambda \in \Lambda} \langle h, X_\lambda \rangle. \tag{a.3}$$

Proof. The proof of the existence part and (a.3) for a non-random h can be found in [10; Appendix]. Clearly, (a.3) remains valid for simple h. Passing on to the limit, we get (a.3) for an arbitrary h. □

References

[1] *P. Barrieu, N. El Karoui.* Pricing, hedging, and optimally designing derivatives via minimization of risk measures. To appear in: R. Carmona (Ed.). Volume on indifference pricing. Princeton.

[2] *P. Carr, H. Geman, D. Madan.* Pricing and hedging in incomplete markets. Journal of Financial Economics, **62** (2001), p. 131–167.

[3] *P. Carr, H. Geman, D. Madan.* Pricing in incomplete markets: from absence of good deals to acceptable risk. In: G. Szegö (Ed.). Risk measures for the 21st century. Wiley, 2004, p. 451–474.

[4] *P. Cheridito, F. Delbaen, M. Kupper.* Dynamic monetary risk measures for bounded discrete-time processes. Electronic Journal of Probability, **11** (2006), p. 57–106.

[5] *P. Cheridito, M. Kupper.* Composition of time-consistent dynamic monetary risk measures in discrete time. Preprint, available at: `http://www.math.ethz.ch/~kupper`.

[6] *P. Cheridito, M. Kupper.* Time-consistency of prices and monetary utility functions. Preprint, available at: `http://www.math.ethz.ch/~kupper`.

[7] *A.S. Cherny.* Pricing with coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[8] *A.S. Cherny.* Equilibrium with coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[9] *A.S. Cherny.* Weighted V@R and its properties. Finance and Stochastics, **10** (2006), No. 3, p. 367–393.

[10] *A.S. Cherny.* Capital allocation and risk contribution with discrete-time coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[11] *A.S. Cherny, D.B. Madan.* Pricing and hedging in incomplete markets with coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[12] *A.S. Cherny, D.B. Madan.* CAPM, rewards, and empirical asset pricing with coherent risk. Preprint, available at: `http://mech.math.msu.su/~cherny`.

[13] *J. Cvitanić, I. Karatzas.* On dynamic measures of risk. Finance and Stochastics, **3** (1999), p. 451–482.

[14] *F. Delbaen.* Coherent monetary utility functions. Preprint, available at `http://www.math.ethz.ch/~delbaen` under the name "Pisa lecture notes".

[15] *N. El Karoui, S. Peng, M.C. Quenez.* Backward stochastic differential equations in finance. Mathematical Finance, **7** (1997), No. 1, p. 1–71.

[16] *H. Föllmer, P. Leukert.* Efficient hedging: cost versus shortfall risk. Finance and Stochastics, **4** (2000), p. 117–146.

[17] *S. Jaschke, U. Küchler.* Coherent risk measures and good deal bounds. Finance and Stochastics, **5** (2001), p. 181–200.

[18] *A. Jobert, L.C.G. Rogers.* Pricing operators and dynamic convex risk measures. Preprint, available at: `http://www.statslab.cam.ac.uk/~chris`.

[19] *S. Klöppel, M. Schweizer.* Dynamic indifference valuation via convex risk measures. Preprint, available at: `http://www.math.ethz.ch/~kloeppel`.

[20] *K. Larsen, T. Pirvu, S. Shreve, R. Tütüncü.* Satisfying convex risk limits by trading. Finance and Stochastics, **9** (2004), p. 177–195.

[21] *J. Leitner.* Pricing and hedging with globally and instantaneously vanishing risk. Preprint.

[22] *Y. Nakano.* Efficient hedging with coherent risk measure. Journal of Mathematical Analysis and Applications, **293** (2004), No. 1, p. 345–354.

[23] *G. Pflug.* Some remarks on the value-at-risk and conditional value-at-risk. In: S. Uryasev (Ed.). Probabilistic constrained optimization: methodology and applications. Kluwer, 2000.

[24] *R.T. Rockafellar, S. Uryasev.* Optimization of conditional Value-At-Risk. Journal of Risk, **2** (2000), No. 3, p. 21–41.

[25] *B. Roorda, J.M. Schumacher, J. Engwerda.* Coherent acceptability measures in multiperiod models. Mathematical Finance, **15** (2005), No. 4, p. 589–612.

[26] *J. Sekine.* Dynamic minimization of worst conditional expectation of shortfall. Mathematical Finance, **14** (2004), No. 4, p. 605–618.

[27] *J. Staum.* Fundamental theorems of asset pricing for good deal bounds. Mathematical Finance, **14** (2004), No. 2, p. 141–161.

[28] *M. Xu.* Risk measure pricing and hedging in incomplete markets. Annals of Finance, **2** (2006), p. 51–71.