ON CERTAIN DISTRIBUTIONS ASSOCIATED WITH THE RANGE OF MARTINGALES*

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Abstract. We study some properties of a continuous local martingale stopped at the first time its range (the difference between the running maximum and minimum) reaches a certain threshold. The laws and the conditional laws of its value, maximum, and minimum at this time are simple and do not depend on the local martingale under question. As a consequence, the price and hedge of options which mature when the range reaches a given level are both model-free within the class of arbitrage-free models with continuous paths, which makes these products very convenient for hedging.

1 Introduction

Let $(S_t)_{t\geq 0}$ be a continuous local martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$. We assume that $\mathcal{F}_0$ is trivial. Consider its running minimum and maximum

$$m_t = \inf_{u\leq t} S_u, \quad M_t = \sup_{u\leq t} S_u, \quad t \geq 0$$

and define the *range process* as

$$R_t = M_t - m_t, \quad t \geq 0.$$

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The problem (which the Reader is invited to solve before reading the paper) is as follows. There are N people sitting around a round table. The people are numbered $1, \dots, N$ clockwise. A plate arrives and the first person gets it. After that, the plate performs a symmetric random walk, i.e. each person receiving the plate transfers it to one of his neighbors with probability 1/2. The random walk stops at the first time everyone has touched the plate. Consider the function

$$p(n) = \mathsf{P}(\text{the } n\text{-th person is the last to touch the plate}), \quad n = 2, \dots, N.$$

The question is at which point/points p attains its maximum.

Define the stopping time

$$\tau = \inf\{t \ge 0 : R_t \ge L\},$$

where $L > 0$ is a given threshold. We will be interested in the unconditional and conditional laws of the random variables S_τ and M_τ. Below we denote by δ_a the Dirac delta-mass concentrated at a point a; by I_A we denote the indicator of a set A. We have the following result; statement (ii) is actually known and can be found in [2; 5.0.4].

Theorem 1.1. *Assume that τ is finite almost surely.*
(i) *The distribution* Law S_τ *is given by*

$$(\text{Law } S_\tau)(dx) = L^{-2}|x - S_0| I_{(S_0 - L, S_0 + L)}(x)dx.$$

(ii) *The distribution* Law M_τ *is given by*

$$(\text{Law } M_\tau)(dx) = L^{-1} I_{(S_0, S_0 + L)}dx.$$

(iii) *The conditional distribution* $\text{Law}(S_\tau \mid \mathcal{F}_t)$ *on the set $\{t < \tau\}$ is given by*

$$\begin{aligned}(\text{Law}(S_\tau \mid \mathcal{F}_t))(dx) &= L^{-1}(M_t - S_t)\delta_{M_t - L}(dx) + L^{-1}(S_t - m_t)\delta_{m_t + L}(dx) \\ &\quad + L^{-2}(S_t - x)I_{(M_t - L, m_t)}(x)dx + L^{-2}(x - S_t)I_{(M_t, m_t + L)}(x)dx.\end{aligned}$$

(iv) *The conditional distribution* $\text{Law}(M_\tau \mid \mathcal{F}_t)$ *on the set $\{t < \tau\}$ is given by*

$$\begin{aligned}(\text{Law}(S_\tau \mid \mathcal{F}_t))(dx) &= L^{-1}(S_t - m_t)\delta_{m_t + L}(dx) + L^{-1}(M_t - S_t)\delta_{M_t}(dx) \\ &\quad + L^{-1} I_{(M_t, m_t + L)}(x)dx.\end{aligned}$$

Remark 1.2. Point (ii) easily follows from (i) if one notes that

$$M_\tau = \begin{cases} S_\tau + L & \text{if } S_\tau \le S_0, \\ S_\tau & \text{if } S_\tau > S_0. \end{cases}$$

Similarly, (iv) is an easy consequence of (iii) since on $\{t < \tau\}$, we have

$$M_\tau = \begin{cases} S_\tau + L & \text{if } S_\tau \le S_t, \\ S_\tau & \text{if } S_\tau > S_t. \end{cases}$$

As $m_\tau = M_\tau - L$, we see that from (i) one can recover the whole $\text{Law}(S_\tau, m_\tau, M_\tau)$, while from (iii) one can recover $\text{Law}(S_\tau, m_\tau, M_\tau \mid \mathcal{F}_t)$.

The above theorem might be given an interesting financial application. Let $(S_t)_{t \ge 0}$ describe the price process of an asset. Imagine an option that pays out the amount $f(S_\tau)$, where τ is the same as above and f is a given function (e.g., $f(x) = (x - K)^+$). Then, as a corollary of the above result, we get the price of this option in any arbitrage-free model with continuous paths. It is given by the theorem below, where we are providing the hedge as well.

Theorem 1.3. *Assume that the risk-free rate is zero and f is integrable on any compact interval. Assume also that $\tau < \infty$ a.s. and the model is arbitrage-free in the sense that there exists an equivalent measure, under which S is a martingale.*[1]

(i) *For any equivalent martingale measure Q, the price $V_0 = \mathsf{E}_{\mathsf{Q}} f(S_\tau)$ is given by*

$$V_0 = L^{-2} \int_{S_0-L}^{S_0+L} |x - S_0| f(x) dx.$$

(ii) *For any equivalent martingale measure Q, the corresponding price process $V_t = \mathsf{E}_{\mathsf{Q}}[f(S_\tau) \mid \mathcal{F}_t]$ is given by*

$$\begin{aligned} V_t &= L^{-1}(M_t - S_t) f(M_t - L) + L^{-1}(S_t - m_t) f(m_t + L) \\ &+ L^{-2} \int_{M_t-L}^{m_t} (S_t - x) f(x) dx + L^{-2} \int_{M_t}^{m_t+L} (x - S_t) f(x) dx \quad \text{on } \{t < \tau\} \end{aligned}$$

and $V_t = f(S_\tau)$ on $\{t \geq \tau\}$.

(iii) *The hedge H is given by*

$$\begin{aligned} H_t = I(t < \tau) \Big[&-L^{-1} f(M_t - L) + L^{-1} f(m_t + L) \\ &+ L^{-2} \int_{M_t-L}^{m_t} f(x) dx - L^{-2} \int_{M_t}^{m_t+L} f(x) dx \Big], \quad t \geq 0, \end{aligned}$$

i.e.

$$V_t = V_0 + \int_0^t H_u dS_u, \quad t \geq 0.$$

We see that the price and the hedge do not depend on the equivalent martingale measure and are thus model-independent within the class of arbitrage-free models with continuous paths. These include, in particular, the Bachelier model, the Black-Scholes-Merton model, the local volatility models as well as the stochastic volatility models. What is more important, the price and the hedge admit a simple analytic form. In those respects, options on the range have similarities with options with the payoff $f(S_\sigma)$, where $\sigma = \inf\{t \geq 0 : \langle S \rangle_t \geq L\}$ (for more information on such options, see [1]).

Let us remark that on the set $\{t \leq \tau\}$, V_t has the form $v(S_t, m_t, M_t)$, while H_t has the form $h(m_t, M_t)$, where

$$h(m, M) = \frac{\partial v(S, m, M)}{\partial S},$$

i.e. H is the delta-hedge. The function v is linear in S, so that h does not depend on S, which means that the hedge remains constant until the price S_t reaches its running maximum or running minimum. This implies that the gamma of the option is zero. However, each time when S breaks through its running maximum or minimum, the hedge is updated (as M_t or m_t changes). Loosely speaking, one can say that the option has a non-zero "right-hand gamma"

$$\frac{\partial^2 v(M + \varepsilon, m, M + \varepsilon)}{\partial \varepsilon_+^2} = L^{-2} f(M) - L^{-2} f(M - L) - L^{-1} f'(M - L)$$

[1] All the results remain the same if the word "martingale" is replaced by "local martingale".

at the time when the price breaks through its running maximum and a non-zero "left-hand gamma"

$$\frac{\partial^2 v(m+\varepsilon, m+\varepsilon, M)}{\partial \varepsilon_-^2} = -L^{-2} f(m+L) + L^{-2} f(m) + L^{-1} f'(m+L)$$

at the time when the price breaks through its running minimum. Here by $\partial^2/\partial\varepsilon_+^2$ (resp., $\partial^2/\partial\varepsilon_-^2$) we denote the right-hand (resp., left-hand) second derivative.

Thus, the hedge should be updated only at the times when the price breaks through its running maximum or minimum. As time grows, these "break points" appear more and more rarely (for example, in a random walk model, the number of such points in the interval $[0, N]$ is of order $N^{1/2}$). This makes the product "quite hedgeable".

Let us finally remark that items (ii) and (iv) of Theorem 1.1 can also be given a financial interpretation. Consider the same setting as before and imagine an option that pays out the amount $f(M_\tau)$. For these options, under the same assumptions as in Theorem 1.4, the following result holds.

Theorem 1.4. **(i)** *For any equivalent martingale measure* Q*, the price* $V_0 = \mathsf{E}_\mathsf{Q} f(M_\tau)$ *is given by*

$$V_0 = L^{-1} \int_{S_0}^{S_0+L} f(x)dx.$$

(ii) *For any equivalent martingale measure* Q*, the corresponding price process* $V_t = \mathsf{E}_\mathsf{Q}[f(M_\tau) \mid \mathcal{F}_t]$ *is given by*

$$V_t = L^{-1}(S_t - m_t) f(m_t + L) + L^{-1}(M_t - S_t) f(M_t) + L^{-1} \int_{M_t}^{m_t+L} f(x)dx \quad \text{on } \{t < \tau\}$$

and $V_t = f(M_\tau)$ *on* $\{t \geq \tau\}$.

(iii) *The hedge* H *is given by*

$$H_t = I(t < \tau)[L^{-1} f(m_t + L) - L^{-1} f(M_t)], \quad t \geq 0.$$

2 Proofs

We will prove Theorems 1.1 and 1.3. Theorem 1.4 is proved in the same way as Theorem 1.3. We are starting with an auxiliary lemma, which is a generalization of the "plate problem" mentioned at the beginning of the paper.

Lemma 2.1. *Let* $(X_n)_{n=0,1,\dots}$ *be a standard symmetric random walk on* $\mathbb{Z}$*. Denote*

$$R_n = \max_{i \leq n} X_i - \min_{i \leq n} X_i, \quad n = 0, 1, \dots$$

and let $\tau = \inf\{n : R_n = N\}$*, where* N *is a fixed natural number. Then*

$$\mathsf{P}(X_\tau = k) = \frac{|k|}{N(N+1)}, \quad k = -N, \dots, N.$$

Proof. Denote $m_n = \min_{i \le n} X_i$. For $k \in \mathbb{Z}$, denote $T_k = \inf\{n : X_n = k\}$. Due to the martingale property of X,

$$\mathsf{P}(m_{T_k} \le l) = \mathsf{P}(T_l < T_k) = \frac{k}{k-l}, \quad l \le 0 < k.$$

Consequently,

$$\mathsf{P}(m_{T_k} = l) = \mathsf{P}(m_{T_k} \le l) - \mathsf{P}(m_{T_k} \le l-1) = \frac{k}{(k-l)(k-l+1)}, \quad l \le 0 < k.$$

As a result,

$$\mathsf{P}(X_\tau = k) = \mathsf{P}(m_{T_k} = k - N) = \frac{k}{N(N+1)}, \quad k = 1, \dots, N.$$

By the symmetry, we get the desired statement for $k = -N, \dots, -1$. □

Proof of Theorem 1.1. In view of Remark 1.2, we have to prove only (i) and (iii).

(i) *Step 1.* Let $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \ge 0}, \mathsf{P}')$ be another probability space with an $(\mathcal{F}'_t)$-Brownian motion B. Consider the enlarged filtered probability space defined by

$$\widetilde{\Omega} = \Omega \times \Omega', \quad \widetilde{\mathcal{F}} = \mathcal{F} \times \mathcal{F}', \quad \widetilde{\mathcal{F}}_t = \mathcal{F}_t \times \mathcal{F}'_t, \quad \widetilde{\mathsf{P}} = \mathsf{P} \times \mathsf{P}'.$$

Then the process

$$\widetilde{S}_t = S_{t \wedge \tau} + \int_0^t I(s \ge \tau) dB_s, \quad t \ge 0$$

is an $(\widetilde{\mathcal{F}}_t, \mathsf{P})$-continuous local martingale that coincides with S up to the time τ and satisfies $\langle S \rangle_\infty = \infty$. It is sufficient to prove the desired statement for $\widetilde{S}$ instead of S. Hence, we can assume from the outset that $\langle S \rangle_\infty = \infty$.

Step 2. For $N \in \mathbb{N}$, define the stopping times $\eta_0^N = 0$,

$$\eta_{n+1}^N = \inf\bigl\{t \ge \eta_n^N : |S_t - S_{\eta_n^N}| = N^{-1}L\bigr\}, \quad n \in \mathbb{N}.$$

Due to the assumption $\langle S \rangle_\infty = \infty$, all the stopping times η_n^N are finite a.s. It follows from the optional stopping theorem that the sequence

$$Y_n^N = S_{\eta_n^N}, \quad n = 0, 1, \dots$$

is a symmetric random walk (multiplied by $N^{-1}L$). Consider

$$\begin{aligned} R_n^N &= \max_{i \le n} Y_i^N - \min_{i \le n} Y_i^N, \quad n = 0, 1, \dots, \\ \sigma^N &= \inf\Bigl\{n : R_n^N = L\,\frac{N-1}{N}\Bigr\}, \\ \tau^N &= \eta_{\sigma^N}^N. \end{aligned}$$

It is easy to see that

$$L\,\frac{N-1}{N} \le R_{\tau^N} \le L.$$

This yields the convergence $S_{\tau^N} \to S_\tau$ a.s. Hence, the convergence in law also holds. Employing Lemma 2.1, we complete the proof.

(iii) Using the same argument as in (i), Step 1, we can assume that $\langle S\rangle_\infty = \infty$. Fix $t \ge 0$ and denote

$$\widetilde{R}_s = \sup_{t\le u\le s} S_u - \inf_{t\le u\le s} S_u, \quad s \ge t,$$
$$\widetilde{\tau} = \inf\{s \ge t : \widetilde{R}_s = L\}.$$

The assumption $\langle S\rangle_\infty = \infty$ ensures that $\widetilde{\tau} < \infty$ a.s. For a.e. $\omega \in \{t < \tau\}$, the conditional distribution $\mathrm{Law}_{\mathsf{Q}}(S_u; u \ge t \mid \mathcal{F}_t)(\omega)$ is the distribution of a continuous martingale. Applying now (i), we see that, for a.e. $\omega \in \{t < \tau\}$, the conditional distribution $\mathsf{Q}_\omega = \mathrm{Law}_{\mathsf{Q}}(S_{\widetilde{\tau}} \mid \mathcal{F}_t)(\omega)$ has the form

$$\mathsf{Q}_\omega(dx) = L^{-2}|x - S_t(\omega)| I_{(S_t(\omega)-L, S_t(\omega)+L)}(x)dx.$$

A direct analysis of the path behavior shows that

$$S_\tau = \begin{cases} m_t + L & \text{if } m_t + L < S_{\widetilde{\tau}} < S_t + L,\\ S_{\widetilde{\tau}} & \text{if } M_t < S_{\widetilde{\tau}} < m_t + L,\\ M_t - L & \text{if } S_t < S_{\widetilde{\tau}} < M_t,\\ m_t + L & \text{if } m_t < S_{\widetilde{\tau}} < S_t,\\ S_{\widetilde{\tau}} & \text{if } M_t - L < S_{\widetilde{\tau}} < m_t,\\ M_t - L & \text{if } S_t - L < S_{\widetilde{\tau}} < M_t - L. \end{cases}$$

This yields the result.

Proof of Theorem 1.3. Statements (i) and (ii) follow directly from Theorem 1.1, so we only have to prove (iii).

Step 1. Suppose that S is a Brownian motion. The process V remains the same if the filtration $(\mathcal{F}_t)$ is replaced by the natural filtration of S, so we can assume from the outset that $(\mathcal{F}_t)$ is the natural filtration of S. According to the representation theorem for Brownian martingales (see [3; Ch. V, Th. 3.4]), there exists a predictable process $(G_t)_{t\ge 0}$ such that

$$V_t = V_0 + \int_0^t G_u dS_u, \quad t \ge 0.$$

Fix $v \ge 0$ and define the stopping time $\sigma_v = \inf\{u \ge v : S_u = m_v \text{ or } S_u = M_v\}$. Consider the processes

$$\widetilde{G}_t = G_t I(t \le \sigma_v), \quad t \ge 0,$$
$$\widetilde{H}_t = H_t I(t \le \sigma_v), \quad t \ge 0,$$
$$\widetilde{V}_t = V_0 + \int_0^t \widetilde{G}_u dS_u, \quad t \ge 0,$$
$$\widetilde{X}_t = V_0 + \int_0^t \widetilde{H}_u dS_u, \quad t \ge 0.$$

On $\{v \le t \le \tau \wedge \sigma_v\}$, the value V_t is an affine function of S_t, and the slope of this function is exactly H_v. Therefore,

$$\widetilde{V}_t - \widetilde{V}_v = H_v(S_t - S_v), \quad v \le t \le \tau \wedge \sigma_v.$$

Obviously, H_t is constant on $\{v \le t \le \tau \wedge \sigma_v\}$, so that

$$\widetilde{X}_t - \widetilde{X}_v = H_v(S_t - S_v), \quad v \le t \le \tau \wedge \sigma_v.$$

As both processes $\widetilde{V}$ and $\widetilde{X}$ are stopped at the time $\tau \wedge \sigma_v$, we see that

$$\widetilde{V}_t - \widetilde{V}_v = \widetilde{X}_t - \widetilde{X}_v, \quad t \ge v.$$

From this we deduce that $\widetilde{G} = \widetilde{H}$ $\mu \times \mathsf{P}$-a.e. on $[v, \infty) \times \Omega$, where μ denotes the Lebesgue measure.

Thus, we have proved that $G = H$ $\mu \times \mathsf{P}$-a.e. on every stochastic interval of the form $\{(\omega, t) : v \le t \le \tau(\omega) \wedge \sigma_v(\omega)\}$. By the definition of H, $H_t = 0$ on $\{t \ge \tau\}$. As V is stopped at the time τ, we can assume from the outset that $G_t = 0$ on $\{t \ge \tau\}$. Thus, $G = H$ $\mu \times \mathsf{P}$-a.e. on every stochastic interval $\{(\omega, t) : v \le t \le \sigma_v(\omega)\}$. Obviously,

$$\{(\omega, t) : m_t(\omega) < S_t(\omega) < M_t(\omega)\} \subseteq \bigcup_{v \in \mathbb{Q}_+} \{(\omega, t) : v \le t \le \sigma_v(\omega)\}. \tag{2.1}$$

It is seen from the form of the joint law of (S_t, M_t) (see [3; Ch. III, Ex. 3.14]) that, for any $t \ge 0$, $\mathsf{P}(S_t = M_t) = 0$; similarly, $\mathsf{P}(S_t = m_t) = 0$. By the Fubini theorem,

$$\mu \times \mathsf{P}\big((\omega, t) : S_t(\omega) = m_t(\omega) \text{ or } S_t(\omega) = M_t(\omega)\big) = \int_0^\infty \mathsf{P}(S_t = m_t \text{ or } S_t = M_t)dt = 0.$$

Thus, the set on the left-hand side of (2.1), and hence, the set on its right-hand side, have a complete $\mu \times \mathsf{P}$-measure. We conclude that $G = H$ $\mu \times \mathsf{P}$-a.e. This yields the desired result.

Step 2. Let us now consider the general case. Without loss of generality, $S_0 = 0$. Define the functions

$$\begin{aligned} g(S, m, M) &= L^{-1}(S - m)f(m + L) - L^{-1}(M - S)f(M - L) \\ &\quad + L^{-2} \int_{M-L}^{m} (S - x)f(x)dx + L^{-2} \int_{M}^{m+L} (x - S)f(x)dx, \\ h(m, M) &= -L^{-1}f(M - L) + L^{-1}f(m + L) + L^{-2} \int_{M-L}^{m} f(x)dx - L^{-2} \int_{M}^{m+L} f(x)dx. \end{aligned}$$

We can assume from the outset that S is stopped at the time τ.

Consider the time change

$$T_t = \inf\{u \ge 0 : \langle S \rangle_u > t\}, \quad t \ge 0,$$

where $\inf \emptyset = \infty$, and define the process $X_t = S_{T_t}$. It follows from [3; Ch. IV, Prop. 1.13] and [3; Ch. V, Prop. 1.5] that the process X is a continuous $(\mathcal{F}_{T_t})$-local martingale with

$$\langle X \rangle_t = \langle S \rangle_{T_t} = t \wedge \langle S \rangle_\tau, \quad t \ge 0.$$

Let R^X denote the range process of X. On $\{\langle S \rangle_\tau > t\}$, we have $T_t < \tau$, and hence, $R_t^X = R_{T_t} < L$; on $\{\langle S \rangle_\tau \le t\}$, we have $T_t = \infty$, and hence, $R_t^X = R_\infty = L$. This shows that the stopping time $\widetilde{\tau} = \inf\{t \ge 0 : R_t^X \ge L\}$ coincides with $\langle S \rangle_\tau$. Thus, $\langle X \rangle_t = t \wedge \widetilde{\tau}$. In particular, we see that X is stopped at the time $\widetilde{\tau}$.

Using the same method as in the proof of Theorem 1.1 (i), we construct (possibly, on an enlarged probability space) a Brownian motion $\widetilde{X}$ that coincides with X up to the time $\widetilde{\tau}$. Then $\widetilde{\tau} = \inf\{t \geq 0 : R_t^{\widetilde{X}} \geq L\}$, where $R^{\widetilde{X}}$ is the range process of $\widetilde{X}$. Thus, X appears as the Brownian motion stopped at the first time its range exceeds L.

Now, it follows from the result of Step 1 that on the set $\{t \leq \widetilde{\tau}\}$ we have

$$g(X_t, m_t^X, M_t^X) = g(0,0,0) + \int_0^t h(m_u^X, M_u^X)dX_u,$$

where m^X and M^X are the running minimum and maximum of X. As X is stopped at the time $\widetilde{\tau}$, we conclude that the above equality is valid for all $t \geq 0$. Thus,

$$\begin{aligned} g(S_{T_t}, m_{T_t}, M_{T_t}) &= g(X_t, m_t^X, M_t^X) \\ &= g(0,0,0) + \int_0^t h(m_u^X, M_u^X)dX_u \\ &= g(0,0,0) + \int_0^t h(m_{T_u}, M_{T_u})dS_{T_u} \\ &= g(0,0,0) + \int_0^{T_t} h(m_u, M_u)dS_u, \quad t \geq 0. \end{aligned} \tag{2.2}$$

The last equality is the time change for stochastic integrals (the combination of [3; Ch. V, Prop. 1.5] with [3; Ch. IV, Prop. 1.13]).

It follows from [3; Ch. IV, Prop. 1.13] that, for a.e. ω, the path $S(\omega)$ is constant on all the intervals of constancy of $\langle S\rangle(\omega)$. Hence, the same is true for m and M. Moreover,

$$\left\langle \int_0^{\cdot} h(m_u, M_u)dS_u \right\rangle_t = \int_0^t h^2(m_u, M_u)d\langle S\rangle_u, \quad t \geq 0.$$

Hence, for a.e. ω, the path of the process $\int_0^{\cdot} h(m_u, M_u)dS_u$ is constant on all the intervals of constancy of $\langle S\rangle(\omega)$. The set of points $\{T_t(\omega); t \geq 0\}$ occupies the whole $\mathbb{R}_+$ except for the intervals of constancy of $\langle S\rangle(\omega)$, but contains all the right endpoints of those intervals. Now, we get from (2.2) the desired result. □

3 Conclusion

We have established the laws of a continuous local martingale and its maximum at a stopping time which is the first time its range reaches a given level. We apply this result to compute the price of options which mature at that stopping time, only under the assumptions of no interest rate, frictionless market, no arbitrage, path continuity, and finiteness of the stopping time. The price is model-free in the sense that it does not depend on the price process. The option is perfectly hedgeable, the hedge is model-free, and it needs rebalancing only when the current minimum or maximum is changed.

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