ISOLATED SINGULAR POINTS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We consider a one-dimensional homogeneous stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x,$$

where b and σ are supposed to be measurable functions and $\sigma \neq 0$. No assumptions of boundedness (or boundedness away from zero) are imposed.

We introduce a class of points called *isolated singular points* and investigate the weak existence as well as the uniqueness in law of the solution in the neighbourhood of such a point. A complete qualitative classification of these points is presented. There are 63 different types. The constructed classification allows us to find out whether a solution can reach an isolated singular point, whether it can leave this point, and so on.

It has been found that, for 59 types, there exists a unique solution in the neighbourhood of the corresponding isolated singular point. (This solution is defined up to the moment it leaves some interval). Moreover, the solution is a strong Markov process.

The remaining 4 types of isolated singular points (we call them *branch types*) disturb the uniqueness. One can construct various "bad" solutions in the neighbourhood of a branch point. In particular, there exist non-Markov solutions.

As the application of the obtained results, we consider equations of the form

$$dX_t = \mu|X_t|^\alpha dt + \nu|X_t|^\beta dB_t, \quad X_0 = x,$$

and present the classification for this case.

Key words and phrases. Stochastic differential equations, singular coefficients, solutions up to a random time, continuous strong Markov processes, local characteristics of a diffusion, speed measure, scale function.

1 Introduction

1. The basis of the theory of *diffusion processes* was formed by Kolmogorov in [16] (the Chapman-Kolmogorov equation, forward and backward partial differential equations). This theory was further developed in a series of papers by Feller (see, for example, [9], [10]). In particular, Feller described the *boundary behaviour* of a diffusion process.

Itô [11], [12] proposed an alternative approach to constructing diffusions. He introduced the notion of a *stochastic differential equation* (abbreviated below as *SDE*). Stroock and Varadhan [22] introduced the concept of a *martingale problem* which is closely connected with the notion of a SDE.

Itô, McKean [13] and Dynkin [4] proposed another approach to the diffusion processes. They proved that a one-dimensional continuous *strong Markov* process that satisfies an additional *regularity* condition can be obtained from a Brownian motion by the following three operations: *random time-change*, *transformation of the phase space* and *killing at a random time*.

The relationship between continuous strong Markov processes and martingale or semimartingale solutions of SDEs is still an interesting problem to be studied. Engelbert and Schmidt proved in [8] that any continuous strong Markov local martingale can be obtained from a solution of a SDE without drift through a special form of *time delay*. Çinlar, Jacod, Protter and Sharpe presented in [3] conditions for a regular continuous strong Markov process to be a semimartingale. Schmidt [21] gave a criterion for a regular continuous strong Markov process to be a solution of a SDE. Similar problems for continuous strong Markov processes with no regularity assumptions were treated by Assing and Schmidt in [1].

2. In this paper, we investigate one-dimensional homogeneous SDEs of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \quad X_0 = x, \tag{1.1}$$

where $(B_t)_{t\geq 0}$ is a standard linear Brownian motion and x is a real number.

We will study the following main problems:

I. *Does there exist a solution of* (1.1) *and is it unique?*

II. *Does it have the strong Markov property?*

III. *What is the qualitative behaviour of the solution?*

Let us first describe the known results related to the existence and uniqueness of solutions of such equations. Most of these results are connected with more general multidimensional inhomogeneous SDEs, i.e., equations of the form

$$dX_t^i = b^i(t, X_t)\,dt + \sum_{j=1}^n \sigma^{ij}(t, X_t)\,dB_t^j, \quad X_0^i = x^i \quad (i = 1, \dots, n). \tag{1.2}$$

The first sufficient condition for the existence and the uniqueness of a solution of (1.2) was obtained by Itô [12]. This condition requires that the coefficients b and σ are locally Lipschitzian.

Stroock and Varadhan [22] proved that there exists a unique solution of (1.2) under the assumption that b is measurable and bounded, while σ is continuous and strictly elliptic. They also proved the following statement. If the coefficients b and σ do not depend on t and, for any x, there exists a unique solution of (1.2), then this solution is a strong Markov process.

Krylov [17], [18] considered multidimensional *homogeneous* SDEs and proved the existence and the uniqueness for the case where b is measurable and bounded, while σ is measurable and strictly elliptic (no continuity assumption on σ was imposed). For the case $n > 2$, an additional assumption was made to guarantee the uniqueness.

The conditions imposed on b and σ in the papers of Stroock, Varadhan, Krylov were much weaker than Itô's condition. Ershov and Shiryaev introduced the notions of *weak* and *strong* solutions (the definitions can be found, for example, in the book [19] by Liptser and Shiryaev). According to this terminology, the solution constructed by Itô is a *strong* solution while the solutions constructed in the later papers (under much weaker assumptions) are *weak* solutions. The relationship between weak and strong solutions was investigated in the paper [23] by Zvonkin and Krylov.

For the special case of one-dimensional homogeneous SDEs (i.e., SDEs of the form (1.1)), there exist much better sufficient conditions for the weak existence and the uniqueness of a solution. This was shown by Engelbert and Schmidt in [5]–[8] (for the case $b = 0$, they gave necessary and sufficient conditions). Engelbert and Schmidt proved that if $\sigma(x) \neq 0$ for any $x \in \mathbb{R}$ and

$$\frac{1 + |b|}{\sigma^2} \in L^1_{\text{loc}}(\mathbb{R}), \tag{1.3}$$

then, for any $x \in \mathbb{R}$, there exists a unique (weak) solution of (1.1).

3. The main goal of this paper is to investigate one-dimensional homogeneous SDEs for which condition (1.3) is violated, i.e., SDEs with *singular coefficients*. The only assumption we make from the outset is that $\sigma(x) \neq 0$ for any $x \in \mathbb{R}$. We will only deal with weak solutions and study the above mentioned Problems I, II, III.

The importance of the stochastic differential equations with singular coefficients both for the theory and for the practical applications can be shown by the following arguments.

There are many examples of SDEs that arise naturally in the stochastic analysis and do not satisfy condition (1.3). Such are, for example, the equations for *Bessel processes* and for the *squares of Bessel processes.*

SDEs with singular coefficients are essential for various applications of the stochastic analysis. Indeed, suppose that we model some process as a solution of SDE (1.1). Assume that this process is positive by its nature (for example, it is the price of a stock on the securities market or the size of a population). Then the SDE used to model such a process should have singular coefficients. The reason is as follows. If condition (1.3) is satisfied, then, for any $x > 0$ and any $a < 0$, the solution started at x reaches the level a with positive probability (see Theorem 7.1 in Section 7).

In order to investigate SDEs with singular coefficients, we introduce the following definition. A point $d \in \mathbb{R}$ is called a *singular* point for SDE (1.1) if

$$\frac{1 + |b|}{\sigma^2} \notin L^1_{\text{loc}}(d)$$

(see Definition 4.1 for the notation $L^1_{\text{loc}}(d)$). According to this definition, any point $d \in \mathbb{R}$ is either a singular point or a regular one. It turns out that there exists a *qualitative* difference between these two classes of points. This difference is expressed in terms of the behaviour of a solution in the neighbourhood of the corresponding point (see Section 4).

Using the above terminology, we can say that a SDE has singular coefficients if and only if the set of its singular points is nonempty. It is worth noting that in practice one often comes across SDEs that have only one singular point (usually, it is zero). Thus, the most important class of singular points is formed by the *isolated singular points*. (We call $d \in \mathbb{R}$ an isolated singular point if d is singular and there exists a deleted neighbourhood of d that consists of regular points).

In this paper, we present a complete qualitative classification of isolated singular points. This classification allows us to make conclusions about the qualitative behaviour of a solution in the neighbourhood of the corresponding point. In particular, the classification allows us to find out whether a solution can reach an isolated singular point and whether it can leave this point. This is done through the coefficients b and σ.

In order to perform this classification, we investigate the behaviour of a solution first in the right-hand neighbourhood of an isolated singular point and then in the left-hand neighbourhood. It has been found that there exist 8 qualitative types of the behaviour of a solution in the one-sided neighbourhood of a point. Therefore, there exist $63 (= 8^2 - 1)$ qualitative types of isolated singular points (see Section 7).

4. This paper is arranged as follows.

Section 2 contains some definitions related to SDEs.

In Section 3, we cite some definitions and statements related to regular continuous strong Markov processes. The behaviour of such a process in the right-hand (left-hand) neighbourhood of a point d may be described by 4 parameters $e_1, \dots, e_4$. These parameters were introduced by Feller [9], Itô and McKean [13]. The parameters show whether the process may leave the point d in the right (left) direction, whether it may reach d from the right (left) side, and so on.

In Section 4, we give the definition of a *singular point*. We prove several statements which show that there exists a qualitative difference between the singular points and the regular ones. This confirms that the given definition of a singular point is reasonable.

In Section 5, we present two examples of a SDE with a singular point. These examples show how a solution may behave in the neighbourhood of such a point. In particular, we investigate the existence and the uniqueness of a solution for SDEs governing Bessel processes.

In Section 6, we define a *solution up to a random time*. This notion is necessary for several reasons (in particular, for treating the explosions). Solutions up to a random time were also considered in [7], [8], [15; Ch. 5, (5.1)].

The most important part of this paper is Section 7. In this section, we investigate the behaviour of a solution in the right-hand neighbourhood of an isolated singular point. We prove that, for any x out of this neighbourhood, there exists a solution defined up to a random time of a special form. Moreover, this solution is a strong Markov process. The local characteristics $e_1, \dots, e_4$ of this process as well as its *speed measure* and *scale function* are expressed by b and σ. This leads to the qualitative classification of *right types* of isolated singular points. It has been found that there are 8 different types. Furthermore, we show that an isolated singular point can have one of 63 possible types. The one-sided classification of isolated singular points is illustrated diagrammatically in Figure 1.

In Section 8, the above classification is applied to the power equations, i.e., equations of the form

$$dX_t = \mu|X_t|^\alpha dt + \nu|X_t|^\beta dB_t, \quad X_0 = x.$$

The right types of zero for this SDE can easily be expressed by μ, ν, α and β (see Figure 2).

2 Stochastic Differential Equations

Definition 2.1. A *solution* of SDE (1.1) is a *pair* (Y, B) of adapted processes on a filtered probability space $\big(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathsf{Q}\big)$ such that

i) B is a $(\mathcal{G}_t, \mathsf{Q})$-Brownian motion (i.e., it is a Brownian motion and a $(\mathcal{G}_t, \mathsf{Q})$-martingale);

ii) for any $t \geq 0$,

$$\int_0^t \big(|b(Y_s)| + \sigma^2(Y_s)\big)\, ds < \infty \quad \mathsf{Q}\text{-a.s.};$$

iii) for any $t \geq 0$,

$$Y_t = x + \int_0^t b(Y_s)\, ds + \int_0^t \sigma(Y_s)\, dB_s \quad \mathsf{Q}\text{-a.s.}$$

Definition 2.2. A solution (Y, B) is called a *strong* solution if the process Y is adapted to the filtration $(\overline{\mathcal{F}}_t^B)$, i.e., the completed natural filtration of B.

A solution in the sense of Definition 2.1 is called a *weak* solution.

Definition 2.3. There is *uniqueness in law* for (1.1) if whenever (Y, B) and $(\widetilde{Y}, \widetilde{B})$ are two solutions (which may be defined on different probability spaces) with the same starting point, then the laws of Y and $\widetilde{Y}$ are equal.

Definition 2.4. There is *pathwise uniqueness* for (1.1) if whenever (Y, B) and $(\widetilde{Y}, B)$ are two solutions on the same filtered probability space with the same starting point, then Y and $\widetilde{Y}$ are indistinguishable.

From here on, it will be more convenient for us to use another definition of a solution (which is equivalent to Definition 2.1). In order to give this definition, we need some notation.

Let $C(\mathbb{R}_+)$ be the space of continuous functions $\mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ = [0, \infty)$. Let $X = (X_t)_{t\geq 0}$ denote the *coordinate process* on $C(\mathbb{R}_+)$, i.e., X is defined by

$$X_t : C(\mathbb{R}_+) \ni \omega \longmapsto \omega(t) \in \mathbb{R}. \tag{2.1}$$

Let $(\mathcal{F}_t)$ be the *canonical filtration* on $C(\mathbb{R}_+)$, i.e., $\mathcal{F}_t = \sigma(X_s;\, s \leq t)$, and $\mathcal{F}$ be the Borel σ-field on $C(\mathbb{R}_+)$. Note that $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$.

Definition 2.5. A *solution* of SDE (1.1) is a *measure* P on $\mathcal{F}$ such that
i) $\mathsf{P}\{X_0 = x\} = 1$;
ii) for any $t \geq 0$,

$$\int_0^t \big(|b(X_s)| + \sigma^2(X_s)\big)\, ds < \infty \quad \mathsf{P}\text{-a.s.};$$

iii) the process

$$M_t = X_t - \int_0^t b(X_s)\, ds \tag{2.2}$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale;
iv) the process

$$M_t^2 - \int_0^t \sigma^2(X_s)\, ds \tag{2.3}$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale.
In what follows, we will call P a *solution started at* x.

`Remarks`. (i) If one accepts Definition 2.5, then the *uniqueness* of a solution does not need a special definition.

(ii) Definitions 2.1 and 2.5 do not cover the case of *exploding* solutions. In Section 6, we give the definition of a *solution up to a random time.* This makes it possible to consider explosions. □

The following statement relates Definition 2.1 and Definition 2.5.

Theorem 2.6. *Suppose that* $\sigma(x) \neq 0$ *for all* $x \in \mathbb{R}$.
(i) *Let* (Y, B) *be a solution of* (1.1) *in the sense of Definition 2.1. Then the measure* $\mathsf{P} = \mathrm{Law}(Y_t;\, t \geq 0)$ *is a solution of* (1.1) *in the sense of Definition 2.5.*
(ii) *Let* P *be a solution of* (1.1) *in the sense of Definition 2.5. Then the pair* (Y, B) *defined by*

$$Y_t = X_t, \quad B_t = \int_0^t \frac{1}{\sigma(X_s)}\, dX_s - \int_0^t \frac{b(X_s)}{\sigma(X_s)}\, ds \tag{2.4}$$

is a solution of (1.1) *on the filtered probability space* $(C(\mathbb{R}_+), \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ *in the sense of Definition 2.1.*

The proof is straightforward.

3 Continuous Strong Markov Processes

We will add an isolated point $\{\Delta\}$ to the real line and consider the functions taking values in $\mathbb{R} \cup \{\Delta\}$.

Definition 3.1. The space $\overline{C}(\mathbb{R}_+)$ consists of the functions $f : \mathbb{R}_+ \to \mathbb{R} \cup \{\Delta\}$ with the following property: there exists a time $\xi(f) \in [0, \infty]$ such that f is continuous on $[0, \xi(f))$ and $f = \Delta$ on $[\xi(f), \infty)$. The time $\xi(f)$ is called the *killing time* of f.

Throughout this section, $X = (X_t)_{t\geq 0}$ denotes the coordinate process on $\overline{C}(\mathbb{R}_+)$, i.e.,

$$X_t : \overline{C}(\mathbb{R}_+) \ni \omega \mapsto \omega(t) \in \mathbb{R} \cup \{\Delta\}; \tag{3.1}$$

$(\mathcal{F}_t)$ will be the canonical filtration on $\overline{C}(\mathbb{R}_+)$, i.e., $\mathcal{F}_t = \sigma(X_s;\ s \leq t)$; $\mathcal{F}$ will stand for the σ-field $\bigvee_{t\geq 0} \mathcal{F}_t = \sigma(X_s;\ s \geq 0)$. Note that $(\mathcal{F}_t)$ is *not* right-continuous. We therefore introduce the filtration $\mathcal{F}_t^+ = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$ $(t \geq 0)$.

Remark. The space $\overline{C}(\mathbb{R}_+)$ may be endowed with a metric that turns it into a *Polish* space. Moreover, the corresponding Borel σ-field coincides with $\mathcal{F}$. The space $C(\mathbb{R}_+)$ is a closed subset of $\overline{C}(\mathbb{R}_+)$ in this metric. □

In the following reasoning, we will use the notations:

$$T_a = \inf\{t \geq 0 : X_t = a\}, \tag{3.2}$$

$$\overline{T}_a = \sup_n \inf\{t \geq 0 : |X_t - a| \leq 1/n\}, \tag{3.3}$$

$$T_{a,b} = T_a \wedge T_b, \tag{3.4}$$

$$\overline{T}_{a,b} = \overline{T}_a \wedge \overline{T}_b. \tag{3.5}$$

Here, $a, b \in \mathbb{R}$. On the set $\{X_0 = \Delta\}$, we have $T_a = \overline{T}_a = \infty$. Note that $T_a \neq \overline{T}_a$ because the process X may be killed just before it reaches a.

Definition 3.2. Let $I \subseteq \mathbb{R}$ be an interval which may be closed, open or semi-open. A *continuous strong Markov process on* I is a family $(\mathsf{P}_x)_{x\in I}$ of probability measures on $\mathcal{F}$ such that

i) for any $x \in I$,

$$\mathsf{P}_x\{X_0 = x\} = 1, \quad \mathsf{P}_x\big\{\forall t \geq 0,\ X_t \in I \cup \{\Delta\}\big\} = 1;$$

ii) for any $A \in \mathcal{F}$, the map $x \mapsto \mathsf{P}_x(A)$ is Borel-measurable;

iii) for any $(\mathcal{F}_t^+)$-stopping time T, any $\mathcal{F}$-measurable nonnegative function Ψ and any $x \in I$,

$$\mathsf{E}_{\mathsf{P}_x}[\Psi \circ \Theta_T | \mathcal{F}_T^+] = \mathsf{E}_{\mathsf{P}_{X_T}}[\Psi] \quad \mathsf{P}_x\text{-a.s.}$$

on the set $\{X_T \neq \Delta\}$. Here, Θ_T is the shift on $\overline{C}(\mathbb{R}_+)$ defined as follows:

$$(\Theta_T \circ X)_t = \begin{cases} X_{t+T} & \text{if } T < \infty, \\ \Delta & \text{if } T = \infty. \end{cases} \tag{3.6}$$

(Obviously, Θ_T is $\mathcal{F}|\mathcal{F}$-measurable).

Definition 3.3. A *regular* continuous strong Markov process on I is a family $(\mathsf{P}_x)_{x\in I}$ that satisfies properties i)–iii) of Definition 3.2 as well as the following conditions:

iv) for any $x \in I$, we have on the set $\{\xi < \infty\}$: $\lim_{t\uparrow\xi} X_t$ exists and does not belong to I P_x-a.s. (here, $\xi = \inf\{t \geq 0 : X_t = \Delta\}$). In other words, X can be killed only at the endpoints of I that do not belong to I;

v) for any $x \in \overset{\circ}{I}$ ($\overset{\circ}{I}$ denotes the interior of I) and any $y \in I$, we have $\mathsf{P}_x\{\exists t \geq 0 : X_t = y\} > 0$.

From here on, we will call regular continuous strong Markov processes simply *regular processes*.

Proposition 3.4. *Suppose that* $(\mathsf{P}_x)_{x\in I}$ *is a regular process. There exists a continuous strictly increasing function* $s:\overset{\circ}{I}\to\mathbb{R}$ *such that* $s(X^{T_{a,b}})$ *is a* P_x*-local martingale for any* $a\le x\le b$ *in* $\overset{\circ}{I}$ *(here,* $X^{T_{a,b}}$ *is the process* X *stopped at* $T_{a,b}$*, where* $T_{a,b}$ *is defined in* (3.4)*). Furthermore, the function* s *is determined uniquely up to an affine transformation, and it satisfies the following property: for any* $a\le x\le b$ *in* I*,*

$$\mathsf{P}_x\{T_b<T_a\}=\frac{s(x)-s(a)}{s(b)-s(a)}.$$

For the proof, see [20; Ch. VII, (3.2)] or [14; (20.7)].

Definition 3.5. A function s with the properties stated in Proposition 3.4 is called the *scale function* of the process $(\mathsf{P}_x)_{x\in I}$.

We now turn to another characteristic of a regular process. For $a\le b$ in I, set

$$G_{a,b}(x,y)=\frac{\bigl(s(x)\wedge s(y)-s(a)\bigr)\bigl(s(b)-s(x)\vee s(y)\bigr)}{s(b)-s(a)},\quad x,y\in[a,b],$$

where s is a variant of the scale function.

Proposition 3.6. *For a regular process* $(\mathsf{P}_x)_{x\in I}$*, there exists a unique measure* m *on* $\overset{\circ}{I}$ *such that, for any nonnegative function* f *and any* $a\le x\le b$ *in* I*,*

$$\mathsf{E}_{\mathsf{P}_x}\Bigl[\int_0^{T_{a,b}}f(X_s)\,ds\Bigr]=2\int_a^b G_{a,b}(x,y)\,f(y)\,m(dy). \tag{3.7}$$

For the proof, see [20; Ch. VII, (3.6)] or [14; (20.10)].

Definition 3.7. The measure m given by Proposition 3.6 is called the *speed measure* of the process $(\mathsf{P}_x)_{x\in I}$.

Remarks. (i) Some authors use the term *speed measure* for $2m$ instead of m.

(ii) The measure m is unique for a fixed choice of the scale function. If another variant of the scale function is taken, then one gets a different G and, as a result, a new m. □

Let $(\mathsf{P}_x)_{x\in I}$ be a continuous strong Markov process and $d\in I\setminus\{r\}$, where r denotes the right endpoint of I. The behaviour of the process (P_x) in the right-hand neighbourhood of d may be described by the following parameters:

$$\begin{aligned}
e_1&=\lim_{x\downarrow d}\mathsf{P}_d\{T_x<\theta\},\\
e_2&=\lim_{y\downarrow d}\lim_{x\downarrow d}\mathsf{P}_x\{T_y<\theta\},\\
e_3&=\lim_{x\downarrow d}\mathsf{P}_x\{T_d<\theta\},\\
e_4&=\lim_{x\downarrow d}\mathsf{P}_x\{\overline{T}_d<\theta\},
\end{aligned}\tag{3.8}$$

where T_d, $\overline{T}_d$ are defined in (3.2), (3.3) and $\theta>0$ is an arbitrary constant.

Proposition 3.8. *The values* $e_1, \dots, e_4$ *do not depend on the choice of* $\theta > 0$. *Moreover, they can form only the following combinations:*

e_1	e_2	e_3	e_4
1	1	1	1
1	1	0	0
0	0	p	1
0	0	0	0
0	1	0	0

Here, p *may take any value from* $[0, 1]$.

For the proof, see [13; §3.3].

Remark. The value $p \in (0, 1)$ corresponds to the case where the process is killed with probability $1 - p$ just before it reaches d. □

4 Isolated Singular Points: The Reasoning

Throughout this section, we assume that $\sigma(x) \neq 0$ for all $x \in \mathbb{R}$. By X we denote the coordinate process on $C(\mathbb{R}_+)$ (see (2.1)).

Definition 4.1. A measurable function $f : \mathbb{R} \to \mathbb{R}$ is *locally integrable at a point* $d \in \mathbb{R}$ if there exists $\delta > 0$ such that

$$\int_{d-\delta}^{d+\delta} |f(x)|\, dx < \infty.$$

We will use the notation: $f \in L^1_{\text{loc}}(d)$.

Definition 4.2. A measurable function f is *locally integrable on a set* $D \subseteq \mathbb{R}$ if f is locally integrable at each point $d \in D$. Notation: $f \in L^1_{\text{loc}}(D)$.

Proposition 4.3. *Suppose that, for SDE* (1.1),

$$\frac{1 + |b|}{\sigma^2} \in L^1_{\text{loc}}(\mathbb{R}).$$

Then, for any $x \in \mathbb{R}$, *there exists a unique solution of* (1.1).

For the proof, see [7] or [8].

Remark. The solution constructed in Proposition 4.3 may explode at a finite time. □

Section 7 contains the following local analog of Proposition 4.3 (see Theorem 7.1). *If the function* $(1 + |b|)/\sigma^2$ *is locally integrable at a point* d, *then there exists a unique solution of* (1.1) *"in the neighbourhood of* d*".* Therefore, such a point should be called "regular".

Definition 4.4. A point $d \in \mathbb{R}$ is called a *singular point* for SDE (1.1) if

$$\frac{1+|b|}{\sigma^2} \notin L^1_{\mathrm{loc}}(d).$$

A point that is not singular will be called *regular*.

Definition 4.5. A point $d \in \mathbb{R}$ is called an *isolated singular point* for (1.1) if d is singular and there exists a deleted neighbourhood of d that consists of regular points.

The next four statements are intended to show that the singular points in the sense of Definition 4.4 are indeed "singular".

Proposition 4.6. *Suppose that* $|b|/\sigma^2 \in L^1_{\mathrm{loc}}(\mathbb{R})$ *and* $1/\sigma^2 \notin L^1_{\mathrm{loc}}(d)$. *Then there exists no solution of* (1.1) *started at* d.

For the proof, see [7] or [8].

Theorem 4.7. *Let* I *be an open interval. Suppose that* $|b|/\sigma^2 \notin L^1_{\mathrm{loc}}(x)$ *for any* $x \in I$. *Then, for any* $x \in I$, *there exists no solution of* (1.1) *started at* x.

`Proof`. Suppose that P is a solution started at $x \in I$. By the occupation times formula (see [20; Ch. VI, (1.6)]) and by the definition of a solution, we have

$$\int_0^t |b(X_s)|\,ds = \int_0^t \frac{|b(X_s)|}{\sigma^2(X_s)}\,d\langle X\rangle_s = \int_{\mathbb{R}} \frac{|b(y)|}{\sigma^2(y)}\,L_t^y(X)\,dy < \infty \quad \mathsf{P}\text{-a.s.} \tag{4.1}$$

Here, $L_t^y(X)$ denotes the local time of X spent in y up to t. As $L_t^y(X)$ is right-continuous in y (see [20; Ch. VI, (1.7)]), we deduce that

$$\mathsf{P}\big\{\forall t \ge 0,\ \forall y \in I,\ L_t^y(X) = 0\big\} = 1.$$

Therefore, for the stopping time $T = 1 \wedge \inf\{t \ge 0 : X_t \notin I\}$, one has

$$\begin{aligned} T &= \int_0^T 1\,ds = \int_0^T \sigma^{-2}(X_s)\,d\langle X\rangle_s \\ &= \int_{\mathbb{R}} \sigma^{-2}(y)\,L_T^y(X)\,dy = \int_I \sigma^{-2}(y)\,L_T^y(X)\,dy = 0. \end{aligned}$$

(We used the fact that $L_T^y(X) = 0$ for $y \notin I$; see [20; Ch. VI, (1.3)]). This leads to a contradiction since $T > 0$. □

Theorem 4.8. *Suppose that* d *is a singular point for* (1.1) *and* P *is a solution started at a point* x. *Then*

$$L_t^d(X) = L_t^{d-}(X) = 0 \quad \mathsf{P}\text{-}a.s.$$

for all $t \ge 0$.

Proof. Since d is a singular point, we have

$$\forall \varepsilon > 0, \ \int_d^{d+\varepsilon} \frac{1+|b(x)|}{\sigma^2(x)}\, dx = \infty \tag{4.2}$$

or

$$\forall \varepsilon > 0, \ \int_{d-\varepsilon}^{d} \frac{1+|b(x)|}{\sigma^2(x)}\, dx = \infty. \tag{4.3}$$

If (4.2) is satisfied, then (4.1), together with the right-continuity of $L_t^y(X)$ in y, guarantees that $\forall t \ge 0,\ L_t^d(X) = 0$ P-a.s. If (4.3) is satisfied, then $\forall t \ge 0,\ L_t^{d-}(X) = 0$ P-a.s.

Let B be the process defined in (2.4). Then

$$\begin{aligned}\int_0^t I(X_s = d)\, dX_s &= \int_0^t I(X_s = d)\, b(X_s)\, ds + \int_0^t I(X_s = d)\, \sigma(X_s)\, dB_s \\ &= \int_0^t I(X_s = d)\, b(X_s)\, ds + M_t,\end{aligned}$$

where M is a $(\mathcal{F}_t, \mathsf{P})$-local martingale (here, $(\mathcal{F}_t)$ is the canonical filtration on $C(\mathbb{R}_+)$). By the occupation times formula (see [20; Ch. VI, (1.6)]),

$$\begin{aligned}\int_0^t I(X_s = d)\, b(X_s)\, ds &= \int_0^t \frac{I(X_s = d)\, b(X_s)}{\sigma^2(X_s)}\, d\langle X\rangle_s \\ &= \int_{\mathbb{R}} \frac{I(x = d)\, b(x)}{\sigma^2(x)}\, L_t^x(X)\, dx = 0 \quad \mathsf{P}\text{-a.s.}\end{aligned}$$

Similarly,

$$\langle M\rangle_t = \int_0^t I(X_s = d)\, \sigma^2(X_s)\, ds = 0 \quad \mathsf{P}\text{-a.s.}$$

Therefore,

$$\int_0^t I(X_s = d)\, dX_s = 0.$$

This equality, combined with the properties of the local times (see [20; Ch. VI, (1.7)]), guarantees that

$$\forall t \ge 0,\ L_t^d(X) = L_t^{d-}(X) \quad \mathsf{P}\text{-a.s.} \tag{4.4}$$

We have already proved that $L_t^d(X) = 0$ or $L_t^{d-}(X) = 0$. This, together with (4.4), leads to the desired statement. □

Theorem 4.9. *Suppose that d is a regular point for SDE* (1.1) *and* P *is a solution started at a point x. Suppose moreover that $\mathsf{P}\{T_d < \infty\} > 0$ (T_d is defined in* (3.2)*). Then, on the set $\{t > T_d\}$, we have*

$$L_t^d(X) > 0, \quad L_t^{d-}(X) > 0 \quad \mathsf{P}\text{-a.s.}$$

This theorem can be derived from Theorem 7.1 in Section 7.

Theorem 4.10. *Suppose that*

$$\frac{1+|b|}{\sigma^2} \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\}), \quad \frac{1+|b|}{\sigma^2} \notin L^1_{\mathrm{loc}}(0).$$

Then there are only 4 possibilities:

1. *There is no solution started at zero.*
2. *There exists a unique solution started at zero, and it is nonnegative (i.e.,* $\mathsf{P}\{\forall t \ge 0,\, X_t \ge 0\} = 1$*).*
3. *There exists a unique solution started at zero, and it is nonpositive.*
4. *There exist a nonnegative solution as well as a nonpositive solution started at zero. In this case, alternating solutions may also exist.*

This theorem follows from the results of Section 7.

Proposition 4.3 and Theorem 4.10 illustrate the qualitative difference between the singular points and the regular ones. If the conditions of Proposition 4.3 are satisfied (in this case, zero is a regular point), then there exists a unique solution P started at zero. Moreover, this solution has alternating signs, i.e.,

$$\mathsf{P}\big\{\forall \varepsilon > 0\ \exists t < \varepsilon : X_t > 0\big\} = 1, \quad \mathsf{P}\big\{\forall \varepsilon > 0\ \exists t < \varepsilon : X_t < 0\big\} = 1.$$

These properties follow from the construction of the solution (see [7], [8]). On the other hand, if the conditions of Theorem 4.10 are satisfied (in this case, zero is an isolated singular point), then the above situation is impossible.

5 Isolated Singular Points: Examples

Throughout this section, X denotes the coordinate process on $C(\mathbb{R}_+)$.

Example 5.1. *For the SDE*

$$dX_t = -\frac{1}{2X_t}\, I(X_t \ne 0)\, dt + dB_t, \quad X_0 = x, \tag{5.1}$$

there exists no solution started at zero.

`Proof`. Suppose that P is a solution started at zero. Let B be the process defined in (2.4). By Itô's formula,

$$\begin{aligned} X_t^2 &= -\int_0^t I(X_s \ne 0)\, ds + 2\int_0^t X_s\, dB_s + t \\ &= \int_0^t I(X_s = 0)\, ds + 2\int_0^t X_s\, dB_s. \end{aligned}$$

By the occupation times formula (see [20; Ch. VI, (1.6)]),

$$\int_0^t I(X_s = 0)\, ds = \int_0^t I(X_s = 0)\, d\langle X\rangle_s = \int_{\mathbb{R}} I(x = 0)\, L_t^x(X)\, dx = 0.$$

Thus, X^2 is a local martingale with $X_0^2 = 0$. Consequently, $X_t^2 = 0$ P-a.s. On the other hand, the measure concentrated on $X \equiv 0$ is not a solution. □

`Remark`. If $x \ne 0$, then (5.1) possesses no solution in the sense of Definition 2.5. However, (5.1) has a solution defined up to the moment $T_0 = \inf\{t \ge 0 : X_t = 0\}$ in the sense of Definition 6.1. Moreover, this solution is unique. □

In the following example, we investigate SDEs for *Bessel processes*.

Example 5.2. *Let us consider the SDE*

$$dX_t = \frac{\delta - 1}{2X_t} I(X_t \neq 0)\, dt + dB_t, \quad X_0 = x, \tag{5.2}$$

with $\delta > 1$, $x \in \mathbb{R}$.

(i) *If* $x \neq 0$ *and* $\delta \geq 2$, *then* (5.2) *has a unique solution.*

(ii) *If* $x = 0$ *or* $1 < \delta < 2$, *then* (5.2) *possesses different solutions.*

Proof. **(i)** With no loss of generality, we may assume that $x > 0$. Let P be the distribution of a δ-dimensional Bessel process started at x. It is well known (see, for example, [20; Ch. XI, §1]) that P is a solution of (5.2) (in the sense of Definition 2.5). Let P' be another solution. Set

$$\mathsf{Q} = \mathrm{Law}(X_t^2;\, t \geq 0 \,|\, \mathsf{P}), \quad \mathsf{Q}' = \mathrm{Law}(X_t^2;\, t \geq 0 \,|\, \mathsf{P}').$$

By Itô's formula, both Q and Q' are solutions of SDE

$$dX_t = \delta dt + 2\sqrt{|X_t|}\, dB_t, \quad X_0 = x^2. \tag{5.3}$$

For this equation, the drift b is constant and the diffusion coefficient σ is Hölder continuous of order $1/2$. Therefore, there is even strong existence and strong uniqueness for (5.3) (see [20; Ch. IX, (3.5)]). By the theorem of Yamada and Watanabe (see [20; Ch. IX, (1.7)]), there is weak uniqueness for (5.3), i.e., $\mathsf{Q}' = \mathsf{Q}$. Hence,

$$\mathrm{Law}(|X_t|;\, t \geq 0 \,|\, \mathsf{P}) = \mathrm{Law}(|X_t|;\, t \geq 0 \,|\, \mathsf{P}'). \tag{5.4}$$

Furthermore, the properties of the Bessel processes guarantee that, for $\delta \geq 2$, $\mathsf{P}\{\forall t \geq 0,\ X_t > 0\} = 1$ (see [20; Ch. XI, §1]). This, together with (5.4), implies that $\mathsf{P}'\{\forall t \geq 0,\ X_t \neq 0\} = 1$. Since the paths of X are continuous and $\mathsf{P}'\{X_0 = x > 0\} = 1$, we get $\mathsf{P}'\{\forall t \geq 0,\ X_t > 0\} = 1$. Using (5.4) once again, we obtain $\mathsf{P} = \mathsf{P}'$.

(ii) We will first suppose that $x = 0$. Let P be defined as above and P' be the image of P under the map

$$C(\mathbb{R}_+) \in \omega \longmapsto -\omega \in C(\mathbb{R}_+).$$

It is easy to verify that P' is also a solution of (5.2) started at zero. The solutions P and P' are different since

$$\mathsf{P}\{\forall t \geq 0,\ X_t \geq 0\} = 1, \quad \mathsf{P}'\{\forall t \geq 0,\ X_t \leq 0\} = 1.$$

Moreover, for any $\alpha \in (0,1)$, the measure $\mathsf{P}^\alpha = \alpha\mathsf{P} + (1-\alpha)\mathsf{P}'$ is also a solution.

Suppose now that $x > 0$. Let P denote the distribution of a δ-dimensional Bessel process started at x. Since $1 < \delta < 2$, we have: $\mathsf{P}\{\exists t > 0 : X_t = 0\} = 1$ (see [20; Ch. XI, §1]). Let P' be the image of P under the map

$$C(\mathbb{R}_+) \ni \omega \longmapsto \omega' \in C(\mathbb{R}_+),$$

$$\omega'(t) = \begin{cases} \omega(t) & \text{if } t \leq T_0(\omega), \\ -\omega(t) & \text{if } t > T_0(\omega). \end{cases}$$

where T_0 is defined in (3.2). Then P' is also a solution of (5.2). □

Remark. If $x = 0$ or $1 < \delta < 2$, then SDE (5.2) possesses different strong solutions as well as solutions that are not strong (see [2]). However, strong solutions are not investigated in this paper. □

6 Solutions up to a Random Time

Throughout this section, X denotes the coordinate process on $\overline{C}(\mathbb{R}_+)$ (see (3.1)) and $(\mathcal{F}_t)$ denotes the canonical filtration.

In what follows, we will need two different definitions: a solution up to T and a solution up to $T-$.

Definition 6.1. Let T be a stopping time on $\overline{C}(\mathbb{R}_+)$. A *solution of* (1.1) *up to* T (or a solution *defined* up to T) is a measure P on $\mathcal{F}_T$ such that

i) $\mathsf{P}\{X_0 = x\} = 1$;

ii) $\int_0^T \big(|b(X_s)| + \sigma^2(X_s)\big)\, ds < \infty$ P-a.s.;

iii) $T < \infty$ P-a.s.;

iv) the process

$$M_t = X_{t\wedge T} - \int_0^{t\wedge T} b(X_s)\, ds$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale;

v) the process

$$M_t^2 - \int_0^{t\wedge T} \sigma^2(X_s)\, ds$$

is a $(\mathcal{F}_t, \mathsf{P})$-local martingale.

In what follows, we will often say that (P, T) is a solution of (1.1) started at x.

`Remarks`. (i) The properties i)–v) imply that $T < \xi$ P-a.s., where $\xi = \inf\{t \ge 0 : X_t = \Delta\}$.

(ii) The measure P is defined on $\mathcal{F}_T$ and not on $\mathcal{F}$ since otherwise it would not be unique. □

We remind that T is called a *predictable* stopping time if there exists an increasing sequence $(T_n)_{n=1}^{\infty}$ of stopping times such that $T_n < T$, $\lim_n T_n = T$. Such a sequence is called a *predicting sequence* for T.

Definition 6.2. Let T be a predictable stopping time on $\overline{C}(\mathbb{R}_+)$ with a predicting sequence (T_n). A *solution of* (1.1) *up to* $T-$ (or a solution *defined* up to $T-$) is a measure P on $\mathcal{F}_{T-}$ such that, for any n, the restriction of P to $\mathcal{F}_{T_n}$ is a solution up to T_n.

In what follows, we will often say that $(\mathsf{P}, T-)$ is a solution of (1.1) started at x.

`Remarks`. (i) Obviously, this definition does not depend on the choice of a predicting sequence for T.

(ii) Definition 6.2 implies that $T \le \xi$ P-a.s. □

Let us now clarify the relationship between the definitions of a solution up to a random time and the standard Definition 2.5.

Theorem 6.3. (i) *Suppose that* $(\mathsf{P}, T-)$ *is a solution of* (1.1) *in the sense of Definition 6.2 and* $T = \infty$ P*-a.s. Then* P *admits a unique extension* $\widetilde{\mathsf{P}}$ *to* $\mathcal{F}$*. Let* Q *be the measure on* $C(\mathbb{R}_+)$ *defined as the restriction of* $\widetilde{\mathsf{P}}$ *to the set* $\{\xi = \infty\}$*. (We use here the obvious property* $\overline{C}(\mathbb{R}_+) \cap \{\xi = \infty\} = C(\mathbb{R}_+)$*). Then* Q *is a solution of* (1.1) *in the sense of Definition 2.5.*

(ii) *Let* Q *be a solution of* (1.1) *in the sense of Definition 2.5. Let* P *be the measure on* $\overline{C}(\mathbb{R}_+)$ *defined as* $\mathsf{P}(A) = \mathsf{Q}(A \cap \{\xi = \infty\})$*. Then* $(\mathsf{P}, \infty-)$ *is a solution of* (1.1) *in the sense of Definition 6.2.*

`Proof.` **(i)** The existence and the uniqueness of $\widetilde{\mathsf{P}}$ follow from the equality

$$\mathcal{F}|\{T = \infty\} = \mathcal{F}_{T-}|\{T = \infty\}.$$

The latter part of the statement as well as **(ii)** are obvious. □

7 The Classification of Isolated Singular Points

We will first investigate the behaviour of a solution of (1.1) in the right-hand neighbourhood of an isolated singular point which is supposed to be equal to zero. A complete qualitative classification in terms of the parameters $e_1, \dots, e_4$ defined in (3.8) is presented.

Throughout this section, we suppose that $\sigma(x) \neq 0$ for all $x \in \mathbb{R}$.

As zero is an isolated singular point, there exists $a > 0$ such that

$$\frac{1 + |b|}{\sigma^2} \in L^1_{\mathrm{loc}}((0, a]). \tag{7.1}$$

We note that the integral

$$\int_0^a \frac{1 + |b(x)|}{\sigma^2(x)}\, dx$$

may converge if zero is an isolated singular point. In this case, the corresponding integral diverges in the left-hand neighbourhood of zero.

A solution P defined up to T will be called *nonnegative* if

$$\mathsf{P}\{\forall t \le T,\ X_t \ge 0\} = 1.$$

We will use the functions

$$\rho(x) = \exp\Big\{\int_x^a \frac{2b(y)}{\sigma^2(y)}\, dy\Big\}, \quad x \in (0, a], \tag{7.2}$$

$$s(x) = \begin{cases} \displaystyle\int_0^x \rho(y)\, dy & \text{if } \displaystyle\int_0^a \rho(y)\, dy < \infty, \\ \displaystyle -\int_x^a \rho(y)\, dy & \text{if } \displaystyle\int_0^a \rho(y)\, dy = \infty \end{cases} \tag{7.3}$$

and the measure

$$m(dx) = \frac{I(0 < x < a)}{\rho(x)\, \sigma^2(x)}\, dx. \tag{7.4}$$

For a stopping time T, we will consider the map

$$\Phi_T : \overline{C}(\mathbb{R}_+) \ni \omega \longmapsto \omega^T \in \overline{C}(\mathbb{R}_+) \tag{7.5}$$

defined as $\omega^T(t) = \omega(t \wedge T(\omega))$ and the map

$$\overline{\Phi}_T : \overline{C}(\mathbb{R}_+) \ni \omega \longmapsto \overline{\omega}^T \in \overline{C}(\mathbb{R}_+) \tag{7.6}$$

defined as

$$\overline{\omega}^T(t) = \begin{cases} \omega(t) & \text{if } t < T(\omega), \\ \Delta & \text{if } t \ge T(\omega). \end{cases}$$

Throughout this section, $e_1, \dots, e_4$ mean the values defined in (3.8).

Theorem 7.1. *Suppose that*

$$\int_0^a \frac{1 + |b(x)|}{\sigma^2(x)}\, dx < \infty.$$

(i) *For any* $x \in [0, a]$*, there exists a unique solution* P_x *of* (1.1) *defined up to* $T_{0,a}$ (*cf.* (3.4)).

(ii) *Set* $\widetilde{\mathsf{P}}_x = \mathsf{P}_x \circ \Phi_{T_{0,a}}^{-1}$ (*cf.* (7.5)). *Then* $(\widetilde{\mathsf{P}}_x)_{x\in[0,a]}$ *is a regular process whose scale function and speed measure are given by* (7.3) *and* (7.4). *Moreover, for this process,*

$$e_1 = 0, \quad e_2 = 0, \quad e_3 = 1, \quad e_4 = 1.$$

If the conditions of Theorem 7.1 are satisfied, we will say that zero has *right type 0*.

`Remark`. Condition iii) of Definition 6.1 guarantees that $T_{0,a}$ is P_x-a.s. finite. Moreover, it follows from (3.7) that $\mathsf{E}_{\mathsf{P}_x} T_{0,a} < \infty$. □

Theorem 7.2. *Suppose that*

$$\int_0^a \rho(x)\, dx < \infty, \quad \int_0^a \frac{1 + |b(x)|}{\rho(x)\, \sigma^2(x)}\, dx < \infty, \quad \int_0^a \frac{|b(x)|}{\sigma^2(x)}\, dx = \infty.$$

(i) *For any* $x \in [0, a]$*, there exists a nonnegative solution* P_x *defined up to* T_a*. Moreover, it is unique in the class of nonnegative solutions.*

(ii) *Set* $\widetilde{\mathsf{P}}_x = \mathsf{P}_x \circ \Phi_{T_a}^{-1}$*. Then* $(\widetilde{\mathsf{P}}_x)_{x\in[0,a]}$ *is a regular process whose scale function and speed measure are given by* (7.3) *and* (7.4). *Moreover, for this process,*

$$e_1 = 1, \quad e_2 = 1, \quad e_3 = 1, \quad e_4 = 1.$$

If the conditions of Theorem 7.2 are satisfied, we will say that zero has *right type 2*.

`Remark`. Under the conditions of Theorem 7.2, we have $\mathsf{E}_{\mathsf{P}_x} T_a < \infty$. This can be derived from a formula similar to (3.7) that is related to regular processes with a reflecting point (see [20; Ch. VII, (3.10)]). □

Theorem 7.3. *Suppose that*

$$\int_0^a \rho(x)\, dx < \infty, \quad \int_0^a \frac{1 + |b(x)|}{\rho(x)\, \sigma^2(x)}\, dx = \infty, \quad \int_0^a \frac{1 + |b(x)|}{\rho(x)\, \sigma^2(x)}\, s(x)\, dx < \infty.$$

(i) *For any solution* (P, T)*, we have* $\mathsf{P}\big\{\forall t \in [T_0, T],\ X_t \le 0\big\} = 1$.

(ii) *For any* $x \in [0, a]$*, there exists a unique solution* P_x *defined up to* $T_{0,a}$.

(iii) *Set* $\widetilde{\mathsf{P}}_x = \mathsf{P}_x \circ \Phi_{T_{0,a}}^{-1}$*. Then* $(\widetilde{\mathsf{P}}_x)_{x\in[0,a]}$ *is a regular process whose scale function and speed measure are given by* (7.3) *and* (7.4). *Moreover, for this process,*

$$e_1 = 0, \quad e_2 = 0, \quad e_3 = 1, \quad e_4 = 1.$$

If the conditions of Theorem 7.3 are satisfied, we will say that zero has *right type 1*.

`Remark`. Statement (i) implies that any solution (P,T) started at $x\le 0$ is nonpositive. □

Theorem 7.4. *Suppose that*

$$\int_0^a \rho(x)\,dx<\infty,\qquad \int_0^a \frac{|b(x)|\,s(x)}{\rho(x)\,\sigma^2(x)}\,dx=\infty,\qquad \int_0^a \frac{s(x)}{\rho(x)\,\sigma^2(x)}\,dx<\infty.$$

(i) *If* (P,T) *is a solution started at* $x>0$*, then* $T<T_0$ P*-a.s.*
(ii) *If* (P,T) *is a solution started at* $x\le 0$*, then* $\mathsf{P}\{\forall t\le T,\ X_t\le 0\}=1$*.*
(iii) *For any* $x\in(0,a)$*, there exists a unique solution* P_x *defined up to* $\overline{T}_{0,a}-$ *(cf.* (3.5)*).*
(iv) *Set* $\overline{\mathsf{P}}_x=\mathsf{P}_x\circ\overline{\Phi}_{\overline{T}_{0,a}}^{-1}$ *(cf.* (7.6)*). Then* $(\overline{\mathsf{P}}_x)_{x\in(0,a)}$ *is a regular process whose scale function and speed measure are given by* (7.3) *and* (7.4)*. Let* $\overline{\mathsf{P}}_0$ *be the measure concentrated on* $X\equiv 0$*. Then* $(\overline{\mathsf{P}}_x)_{x\in[0,a)}$ *is a continuous strong Markov process with*

$$e_1=0,\quad e_2=0,\quad e_3=0,\quad e_4=1.$$

If the conditions of Theorem 7.4 are satisfied, we will say that zero has *right type 6*.

Theorem 7.5. *Suppose that*

$$\int_0^a \rho(x)\,dx<\infty,\qquad \int_0^a \frac{s(x)}{\rho(x)\,\sigma^2(x)}\,dx=\infty.$$

(i) *If* (P,T) *is a solution started at* $x>0$*, then* $T<T_0$ P*-a.s.*
(ii) *If* (P,T) *is a solution started at* $x\le 0$*, then* $\mathsf{P}\{\forall t\le T,\ X_t\le 0\}=1$*.*
(iii) *For any* $x\in(0,a)$*, there exists a unique solution* P_x *defined up to* $\overline{T}_a-$*. Moreover,* $\mathsf{P}_x\{\overline{T}_a=\infty \text{ and } X_t\xrightarrow[t\to\infty]{}0\}>0$*.*
(iv) *Set* $\overline{\mathsf{P}}_x=\mathsf{P}_x\circ\overline{\Phi}_{\overline{T}_a}^{-1}$*. Then* $(\overline{\mathsf{P}}_x)_{x\in(0,a)}$ *is a regular process whose scale function and speed measure are given by* (7.3) *and* (7.4)*. Let* $\overline{\mathsf{P}}_0$ *be the measure concentrated on* $X\equiv 0$*. Then* $(\overline{\mathsf{P}}_x)_{x\in[0,a)}$ *is a continuous strong Markov process with*

$$e_1=0,\quad e_2=0,\quad e_3=0,\quad e_4=0.$$

If the conditions of Theorem 7.5 are satisfied, we will say that zero has *right type 4*.

Theorem 7.6. *Suppose that*

$$\int_0^a \rho(x)\,dx=\infty,\qquad \int_0^a \frac{1+|b(x)|}{\rho(x)\,\sigma^2(x)}\,|s(x)|\,dx<\infty.$$

(i) *If* (P,T) *is a solution started at* $x>0$*, then* $T<T_0$ P*-a.s.*
(ii) *For any* $x\in(0,a]$*, there exists a unique solution* P_x *defined up to* T_a*. For* $x=0$*, there exists a nonnegative solution* P_0 *defined up to* T_a*, and it is unique in the class of nonnegative solutions.*
(iii) *Set* $\widetilde{\mathsf{P}}_x=\mathsf{P}_x\circ\Phi_{T_a}^{-1}$*. Then* $(\widetilde{\mathsf{P}}_x)_{x\in(0,a]}$ *is a regular process whose scale function and speed measure are given by* (7.3) *and* (7.4)*. Moreover,* $(\widetilde{\mathsf{P}}_x)_{x\in[0,a]}$ *is a continuous strong Markov process with*

$$e_1=1,\quad e_2=1,\quad e_3=0,\quad e_4=0.$$

If the conditions of Theorem 7.6 are satisfied, we will say that zero has *right type 3*.

Theorem 7.7. *Suppose that*

$$\int_0^a \rho(x)\,dx = \infty, \qquad \int_0^a \frac{|b(x)\,s(x)|}{\rho(x)\,\sigma^2(x)}\,dx = \infty, \qquad \int_0^a \frac{|s(x)|}{\rho(x)\,\sigma^2(x)}\,dx < \infty.$$

(i) *If* (P,T) *is a solution started at* $x > 0$*, then* $T < T_0$ P*-a.s.*
(ii) *If* (P,T) *is a solution started at* $x \le 0$*, then* $\mathsf{P}\bigl\{\forall t \le T,\ X_t \le 0\bigr\} = 1$*.*
(iii) *For any* $x \in (0,a]$*, there exists a unique solution* P_x *defined up to* T_a*.*
(iv) *Set* $\widetilde{\mathsf{P}}_x = \mathsf{P}_x \circ \Phi_{T_a}^{-1}$*. Then* $(\widetilde{\mathsf{P}}_x)_{x\in(0,a]}$ *is a regular process whose scale function and speed measure are given by* (7.3) *and* (7.4)*. Let* $\widetilde{\mathsf{P}}_0$ *be the measure concentrated on* $X \equiv 0$*. Then* $(\widetilde{\mathsf{P}}_x)_{x\in[0,a]}$ *is a continuous strong Markov process with*

$$e_1 = 0, \quad e_2 = 1, \quad e_3 = 0, \quad e_4 = 0.$$

If the conditions of Theorem 7.7 are satisfied, we will say that zero has *right type 7*.

Theorem 7.8. *Suppose that*

$$\int_0^a \rho(x)\,dx = \infty, \qquad \int_0^a \frac{|s(x)|}{\rho(x)\,\sigma^2(x)}\,dx = \infty.$$

(i) *If* (P,T) *is a solution started at* $x > 0$*, then* $T < T_0$ P*-a.s.*
(ii) *If* (P,T) *is a solution started at* $x \le 0$*, then* $\mathsf{P}\bigl\{\forall t \le T,\ X_t \le 0\bigr\} = 1$*.*
(iii) *For any* $x \in (0,a]$*, there exists a unique solution* P_x *defined up to* T_a*.*
(iv) *Set* $\widetilde{\mathsf{P}}_x = \mathsf{P}_x \circ \Phi_{T_a}^{-1}$*. Then* $(\widetilde{\mathsf{P}}_x)_{x\in(0,a]}$ *is a regular process whose scale function and speed measure are given by* (7.3) *and* (7.4)*. Let* $\widetilde{\mathsf{P}}_0$ *be the measure concentrated on* $X \equiv 0$*. Then* $(\widetilde{\mathsf{P}}_x)_{x\in[0,a]}$ *is a continuous strong Markov process with*

$$e_1 = 0, \quad e_2 = 0, \quad e_3 = 0, \quad e_4 = 0.$$

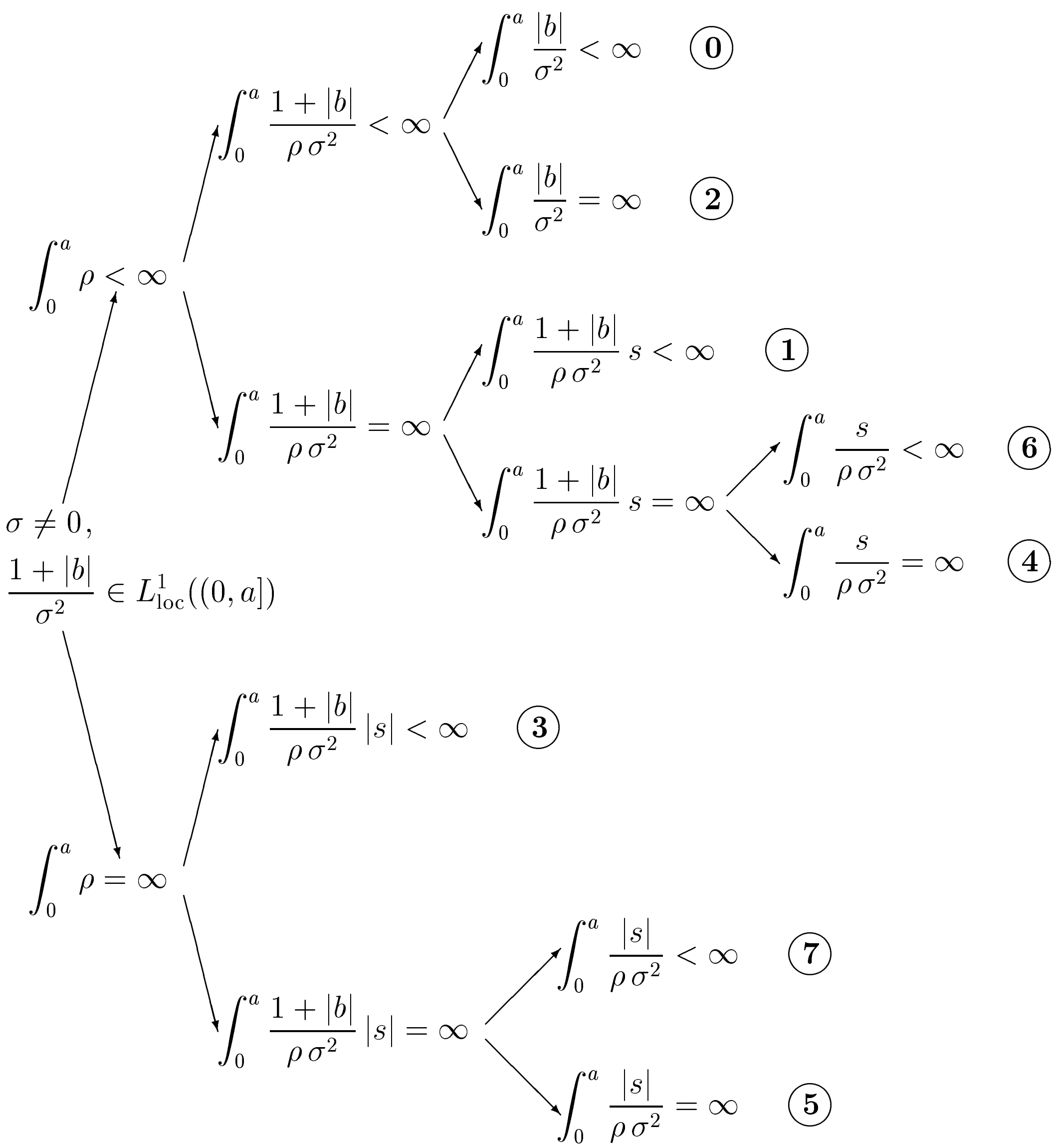
If the conditions of Theorem 7.8 are satisfied, we will say that zero has *right type 5*.
For the sake of brevity, we present here only the statements of the results and give no proofs. The proofs can be found in the forthcoming paper by the same authors.
Figure 1 represents the one-sided classification of isolated singular points. We note that the integrability conditions given on Figure 1 do not have the same form as those given by Theorems 7.1–7.8. Nevertheless, they are equivalent. For example,

$$\int_0^a \frac{1+|b(x)|}{\sigma^2(x)}\,dx < \infty$$

if and only if

$$\int_0^a \rho(x)\,dx < \infty, \qquad \int_0^a \frac{1+|b(x)|}{\rho(x)\,\sigma^2(x)}\,dx < \infty, \qquad \int_0^a \frac{|b(x)|}{\sigma^2(x)}\,dx < \infty.$$

(In this case zero has right type 0).
If zero has right type 2 or 3, then there exist positive solutions started at zero. Thus, types 2 and 3 may be called *entrance* types. On the other hand, types 1, 4, 5, 6, 7 are *non-entrance* ones: any solution started at zero is nonpositive.



Type	e_1	e_2	e_3	e_4
0	0	0	1	1
1	0	0	1	1
2	1	1	1	1
3	1	1	0	0
4	0	0	0	0
5	0	0	0	0
6	0	0	0	1
7	0	1	0	0

Types	Exit	Non-exit
Entrance	**2**	**3**
Non-entrance	**0 1**	**4 5 6 7**

Figure 1. One-sided classification of isolated singular points

If zero has right type 0, 1 or 2, then, for $x \in (0, a)$, there exists a solution started at x that reaches zero with positive probability. Thus, types 0, 1, 2 may be called *exit* types. If the right type of zero is one of $3, \dots, 7$, then any solution with $x > 0$ does not reach zero. So, these types are *non-exit* ones.

Let us now informally describe how a solution behaves in the right-hand neighbourhood of an isolated singular point for each of the types $0, \dots, 7$.

If zero has right type **0**, then, for any $x \in [0, a]$, there exists a unique solution defined up to $T_{0,a}$. This solution reaches zero with positive probability. An example of a SDE for which zero has right type 0 is provided by the equation

$$dX_t = dB_t, \quad X_0 = x.$$

If zero has right type **1**, then, for any $x \in [0, a]$, there exists a unique solution defined up to $T_{0,a}$. This solution reaches zero with positive probability. Any solution started at zero (it may be defined up to another stopping time) is nonpositive. In other words, a solution may leave zero only in the negative direction. The SDE for the square of a 0-dimensional Bessel process

$$dX_t = 2\sqrt{|X_t|}\, dB_t, \quad X_0 = x$$

provides an example of a SDE for which zero has right type 1.

If zero has right type **2**, then, for any $x \in [0, a]$, there exists a unique *nonnegative* solution defined up to T_a. This solution reaches zero with positive probability and is reflected at this point. There may exist other solutions up to T_a (these solutions may take negative values). For the SDE

$$dX_t = \frac{\delta - 1}{2|X_t|} I(X_t \neq 0)\, dt + dB_t, \quad X_0 = x$$

with $1 < \delta < 2$ (this is the SDE for a δ-dimensional Bessel process), zero has right type 2.

If zero has right type **3**, then, for any $x \in (0, a]$, there exists a unique solution defined up to T_a. This solution never reaches zero. There exists a unique *nonnegative* solution started at $x = 0$ and defined up to T_a (for $x = 0$, there may exist other solutions that take negative values and are defined up to T_a). For the SDE

$$dX_t = \frac{\delta - 1}{2|X_t|} I(X_t \neq 0)\, dt + dB_t, \quad X_0 = x$$

with $\delta \geq 2$, zero has right type 3.

If zero has right type **4**, then, for any $x \in (0, a)$, there exists a unique solution defined up to $\overline{T}_a-$. This solution never reaches zero. There exists no solution up to T_a for the following reason. The above mentioned solution tends to zero with positive probability as $t \to \infty$. So, this solution never reaches the point a with positive probability. On the other hand, if (P, T_a) is a solution, then T_a should be P-a.s. finite. For type 4 as well as for types 5, 6, 7 below, any solution started at zero is nonpositive. An example of a SDE for which zero has right type 4 is provided by the equation

$$dX_t = \frac{1}{3}|X_t|\, dt + |X_t|\, dB_t, \quad X_0 = x.$$

If zero has right type **5**, then, for any $x \in (0, a]$, there exists a unique solution defined up to T_a. This solution never reaches zero. As opposed to the previous case, the solution reaches the point a a.s. For the SDE

$$dX_t = \frac{1}{2}|X_t|\,dt + |X_t|\,dB_t, \quad X_0 = x,$$

zero has right type 5.

If zero has right type **6**, then, for any $x \in (0, a)$, there exists a unique solution P_x defined up to $\overline{T}_{0,a}-$. Moreover, $\overline{T}_{0,a}$ is finite P_x-a.s. However, there exists no solution up to $T_{0,a}$ because the integral $\int_0^{\overline{T}_{0,a}} |b(X_s)|\,ds$ equals infinity with positive probability.

If zero has right type **7**, then the qualitative behaviour of a solution is almost the same as for right type 5. The only difference is in the value e_2. We do not give the examples of SDEs for which zero has right type 6 or 7 because these examples are rather complicated.

So far, we have investigated the behaviour of a solution in the right-hand neighbourhood of zero (recall that zero is assumed to be an isolated singular point). According to the classification given above, zero has one of 8 possible right types. If zero has right type 0, then

$$\exists a > 0 : \int_0^a \frac{1 + |b(x)|}{\sigma^2(x)}\,dx < \infty. \tag{7.7}$$

If the right type of zero is one of $1, \dots, 7$, then

$$\forall a > 0, \int_0^a \frac{1 + |b(x)|}{\sigma^2(x)}\,dx = \infty$$

(this can easily be seen from Figure 1).

In a similar way, one can define *left types* of zero (there are 8 left types). We will say that zero *has type* $(i–j)$ if it has left type i and right type j. Thus, there are $64 (= 8 \times 8)$ possibilities. If zero has type $(0 - 0)$, then, in view of (7.7), $(1 + |b|)/\sigma^2 \in L^1_{\mathrm{loc}}(0)$ and so, zero is not a singular point. For the other 63 possibilities, this function is not locally integrable at zero. As a result, an isolated singular point can have one of 63 possible types.

It is easy to see that only 4 of these 63 types can disturb the uniqueness of a solution. These are types $(2-2)$, $(2-3)$, $(3-2)$ and $(3-3)$. Indeed, if zero has one of these types, then there exist both nonnegative and nonpositive solutions started at zero. Therefore, we call these 4 types the *branch types* while corresponding isolated singular points are called the *branch points*. If SDE (1.1) has a branch point, then one can easily construct non-Markov solutions. We will illustrate this by the following example.

Example 7.9. *Let us consider the SDE*

$$dX_t = \frac{\delta - 1}{2|X_t|} I(X_t \neq 0)\,dt + dB_t, \quad X_0 = x \tag{7.8}$$

with $1 < \delta < 2$*. Take* $x > 0$ *and let* P *be the nonnegative solution started at* x *(this is the distribution of the* δ*-dimensional Bessel process started at* x*). Let us consider the map*

$$C(\mathbb{R}_+) \ni \omega \longmapsto \omega' \in C(\mathbb{R}_+)$$

defined as

$$\omega'(t) = \begin{cases} \omega(t) & \text{if } t \le T_0(\omega), \\ \omega(t) & \text{if } t > T_0(\omega) \text{ and } \omega(T_0(\omega)/2) > 1, \\ -\omega(t) & \text{if } t > T_0(\omega) \text{ and } \omega(T_0(\omega)/2) \le 1. \end{cases}$$

Then the image P' *of* P *under this map is a non-Markov solution of* (7.8).

8 Application to Power Equations

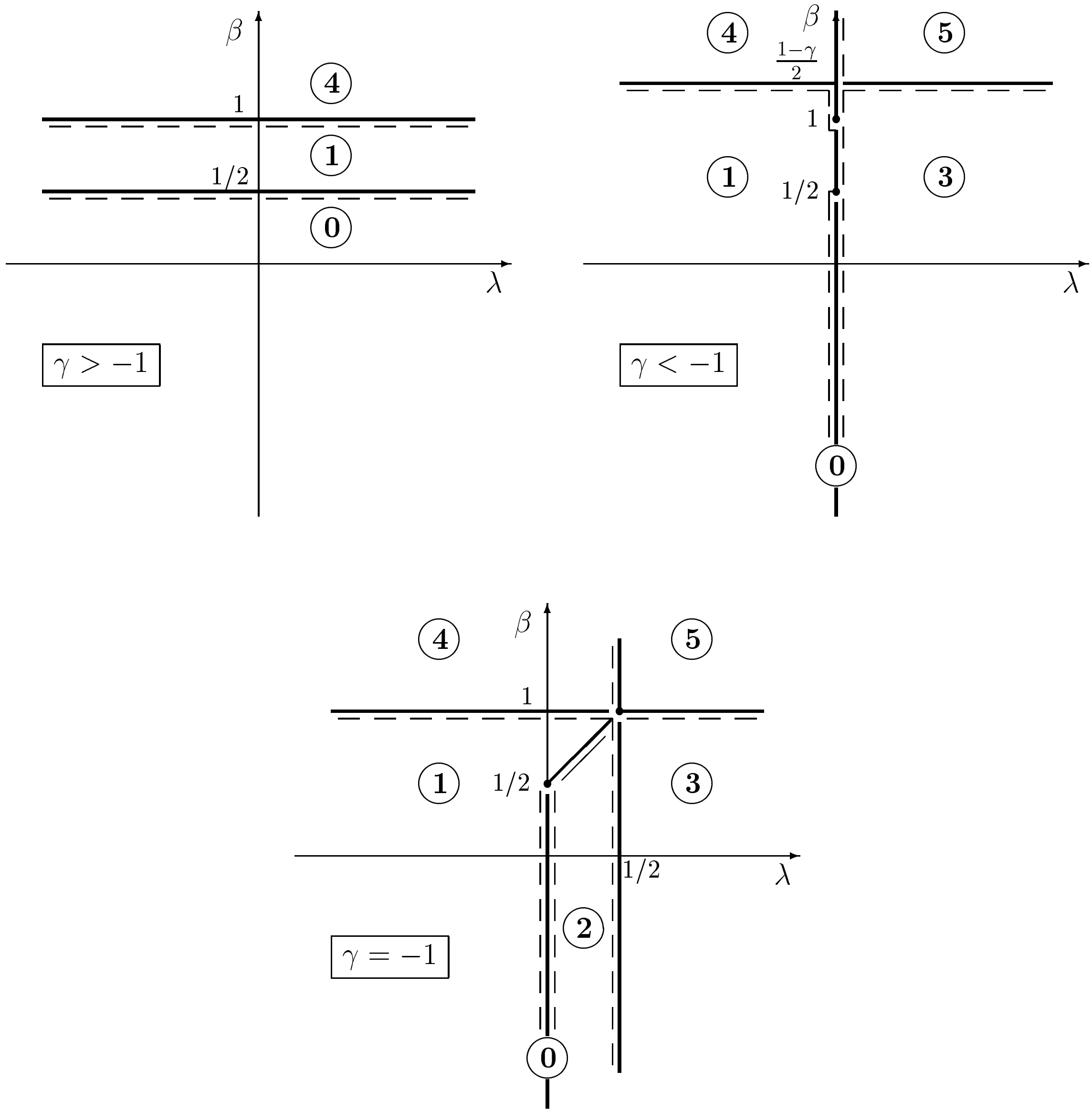
Let us consider the SDE

$$dX_t = \mu\,|X_t|^\alpha\,I(X_t \ne 0)\,dt + \big(\nu\,|X_t|^\beta\,I(X_t \ne 0) + \eta\,I(X_t = 0)\big)\,dB_t. \tag{8.1}$$

Here, $\nu \ne 0$, $\eta \ne 0$. Obviously, all the properties of (8.1) are the same for any $\eta \ne 0$. We add the term $\eta\,I(X_t \ne 0)$ to guarantee that $\sigma \ne 0$ at each point.

Theorem 8.1. *Set* $\lambda = \mu/\nu^2$, $\gamma = \alpha - 2\beta$. *Then right types of zero for* (8.1) *are those given in Figure 2.*

The proof of this statement easily follows from the classification of the right types given above.



`Figure 2.` One-sided classification for the power equations. Here, $\lambda = \mu/\nu^2$, $\gamma = \alpha - 2\beta$, where α, β, μ and ν are given by (8.1). One should first calculate γ and select the corresponding graph (out of the three graphs shown). Then one should plot the point (λ, β) on this graph and find the part of the graph the point lies in. The number ⓘ marked in this part indicates that, for equation (8.1), zero has right type i. For example, if $\gamma < -1$, $\lambda > 0$ and $\beta \geq (1-\gamma)/2$, then zero has right type 5.

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