To our Teacher A.N. Shiryaev
on the occasion of his 70th birthday

ON THE ABSOLUTE CONTINUITY AND SINGULARITY OF MEASURES ON FILTERED SPACES: SEPARATING TIMES

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Abstract. We introduce the notion of a *separating time* for a pair of measures P and $\widetilde{\mathsf{P}}$ on a filtered space. This notion is convenient for describing the mutual arrangement of P and $\widetilde{\mathsf{P}}$ from the viewpoint of their absolute continuity and singularity.

Furthermore, we find the explicit form of the separating time for the cases, where P and $\widetilde{\mathsf{P}}$ are distributions of Lévy processes, solutions of stochastic differential equations, and distributions of Bessel processes. The obtained results yield, in particular, criteria for the local absolute continuity, absolute continuity, and singularity of P and $\widetilde{\mathsf{P}}$.

Key words and phrases. Absolute continuity, Bessel processes, Lévy processes, local absolute continuity, separating times, singularity, stochastic differential equations.

Contents

1 Introduction

The problems of absolute continuity and singularity of probability measures defined on a filtered space play a significant role both in the pure stochastic analysis and in its applications (for example, financial mathematics). The contribution of A.N. Shiryaev to this subject is large and well known. This is represented, in particular, by his papers [13], [14], [22], [23], [24], [25], [28] as well as his monographs [12], [26], [27], and [37]. The plenary talk of A.N. Shiryaev at the International Congress of Mathematics (Helsinki, 1978) was entitled "Absolute continuity and singularity of probability measures in functional spaces". We therefore hold it an honor to be able to put our paper in the Festschrift.

The problems that are typically studied in relation to the subject mentioned concern such questions as: whether two measures are (locally) absolutely continuous, whether they are singular, etc. However, a situation may naturally occur, where the two measures are neither (locally) absolutely continuous nor singular. Consider the following example: $\Omega = C([0,\infty))$, $(\mathcal{F}_t)$ is the canonical filtration, and P (resp., $\widetilde{\mathsf{P}}$) is the distribution of a γ-dimensional (resp., $\widetilde{\gamma}$-dimensional) Bessel process started at a point $x_0 > 0$. If $\gamma \wedge \widetilde{\gamma} < 2$, then, for any $t > 0$, the measures $\mathsf{P}_t = \mathsf{P} \mid \mathcal{F}_t$ and $\widetilde{\mathsf{P}}_t = \widetilde{\mathsf{P}} \mid \mathcal{F}_t$ are neither equivalent nor singular. To be more precise, the situation is as follows: for any stopping time τ such that $\tau < T_0 := \inf\{t \geq 0 : X_t = 0\}$ (here X is the coordinate process), the measures $\mathsf{P}_\tau = \mathsf{P} \mid \mathcal{F}_\tau$ and $\widetilde{\mathsf{P}}_\tau = \widetilde{\mathsf{P}} \mid \mathcal{F}_\tau$ are equivalent; for any stopping time $\tau \geq T_0$, P_τ and $\widetilde{\mathsf{P}}_\tau$ are singular. Thus, the time T_0 plays the following important role in this example: informally, this is the time, at which P and $\widetilde{\mathsf{P}}$ are separated one from another.

The situation described above admits a clear interpretation in terms of statistical sequential analysis, which is another big topic of the research activity of A.N. Shiryaev (this is reflected, in particular, by his monographs [27], [36], and [38]). Suppose that we are observing a process X that is governed either by the measure P or by the measure $\widetilde{\mathsf{P}}$ (these are the measures described above) and are trying to distinguish between these two hypotheses. Then, until the time X hits zero, we cannot say for sure what the true measure is; but immediately after this time we can say for sure what the true measure is. This situation is in contrast with the typical setup of statistical sequential analysis, where the two hypotheses are typically assumed to be locally equivalent.

Let us now consider the general situation: let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a space with a right-continuous filtration (here $\mathcal{F} = \bigvee_t \mathcal{F}_t$) and P, $\widetilde{\mathsf{P}}$ be two probability measures on this space. In Section 2, we formalize the concept of the time, at which the two measures are separated. Namely, we prove that there exists a $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. unique stopping time S with the property: for any stopping time τ, the measures P_τ and $\widetilde{\mathsf{P}}_\tau$ are equivalent on the set $\{\tau < S\}$ and are singular on the set $\{\tau \ge S\}$ (actually, S is given by $\inf\{t \ge 0 : Z_t = 0 \text{ or } Z_t = 2\}$, where Z denotes the density process of P with respect to $(\mathsf{P} + \widetilde{\mathsf{P}})/2$). Informally, P and $\widetilde{\mathsf{P}}$ are equivalent before the time S and are singular after this time. We call S the *separating time for* P *and* $\widetilde{\mathsf{P}}$. In order to be able to distinguish the situation, where P and $\widetilde{\mathsf{P}}$ are locally equivalent and are globally singular (i.e. singular on $\mathcal{F}$), from the situation, where they are globally equivalent, we add a point $\delta > \infty$ to $[0, \infty]$ and allow S to take values in $[0, \infty] \cup \{\delta\}$ (informally, the equality $S(\omega) = \delta$ means that P and $\widetilde{\mathsf{P}}$ are "globally equivalent on the elementary outcome ω"). The properties such as (local) absolute continuity and singularity are easily expressed in terms of a separating time (see Lemma 2.7). For example, $\widetilde{\mathsf{P}} \ll \mathsf{P}$ iff $S = \delta$ $\widetilde{\mathsf{P}}$-a.s., $\widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P}$ iff $S \ge \infty$ $\widetilde{\mathsf{P}}$-a.s. (i.e. $\widetilde{\mathsf{P}}(S \in \{\infty, \delta\}) = 1$); $\widetilde{\mathsf{P}}_0 \perp \mathsf{P}_0$ iff $S = 0$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s., etc.

In order to illustrate the notion of a separating time, we give in Section 3 the explicit form of this time for the case, where P and $\widetilde{\mathsf{P}}$ are distributions of Lévy processes. This is just a translation of known results into the language of separating times.

In Section 4, we consider the case, where P and $\widetilde{\mathsf{P}}$ are distributions of Bessel processes of different dimensions started at the same point and prove that in this case the separating time has the form $S = \inf\{t \ge 0 : X_t = 0\}$, where X denotes the coordinate process. This puts the above discussion related to Bessel processes on a solid mathematical basis.

The introduction of separating times enables us to give a complete answer to the problem of (local) absolute continuity and singularity of solutions of one-dimensional homogeneous stochastic differential equations (abbreviated below as SDEs), i.e. equations of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0 \tag{1.1}$$

(the conditions we impose on the coefficients are the Engelbert–Schmidt conditions, i.e. b and σ are measurable, $\sigma \ne 0$ pointwise, and $(1 + |b|)/\sigma^2 \in L^1_{\text{loc}}(\mathbb{R})$; this guarantees the existence and the uniqueness of a solution). Namely, in Section 5, we find the explicit form of the separating time for the measure P being the solution of (1.1) and the measure $\widetilde{\mathsf{P}}$ being the solution of a SDE

$$dX_t = \widetilde{b}(X_t)dt + \widetilde{\sigma}(X_t)dB_t, \quad X_0 = x_0.$$

As a corollary, we obtain criteria for (local) absolute continuity and singularity of P and $\widetilde{\mathsf{P}}$. The problems of (local) absolute continuity and singularity for diffusion processes were extensively studied earlier. Let us mention the papers [8], [10], [13], [14], [15], [16], [17], [23], [31] and the monographs [12; Ch. IV, § 4b], [27; Ch. 7]. We consider here a more particular case (only homogeneous SDEs), but in this case we obtain more complete results. Namely, in the majority of the sources mentioned above, conditions for (local) absolute continuity and singularity are given in random terms (typically, in terms of the Hellinger process). In contrast, here the explicit form of the separating time and conditions for (local) absolute continuity and singularity are obtained in nonrandom terms, i.e. in terms of the coefficients of SDEs. In this respect, our results are similar to those in [31]. Furthermore, all the sources mentioned above (including [31]) deal with (local) absolute continuity or singularity of measures, while our results are applicable to measures that are in a general position, i.e. they are neither (locally) equivalent nor singular.

Let us illustrate the structure of the results of Section 5 by a simple example. Let P and $\widetilde{\mathsf{P}}$ be solutions of SDEs

$$\begin{aligned} dX_t &= \sigma(X_t)dB_t, \quad X_0 = x_0, \\ dX_t &= \widetilde{b}(X_t)dt + \widetilde{\sigma}(X_t)dB_t, \quad X_0 = x_0, \end{aligned}$$

respectively. We assume that both equations satisfy the Engelbert–Schmidt conditions. Let us also assume for the simplicity of presentation that $\widetilde{\mathsf{P}}$ is nonexploding (P is nonexploding automatically), although we consider exploding solutions as well. Our results yield that the separating time for P and $\widetilde{\mathsf{P}}$ has the form:

$$S = \begin{cases} \delta & \text{if } \widetilde{b} = 0 \text{ and } \widetilde{\sigma}^2 = \sigma^2 \ \ \mu_L\text{-a.e.}, \\ \inf\{t \ge 0 : X_t \in A\} & \text{otherwise}, \end{cases}$$

where X denotes the coordinate process, $\inf \emptyset := \infty$, μ_L denotes the Lebesgue measure, and A denotes the complement to the set

$$\big\{x \in \mathbb{R} : \widetilde{b}^2/\widetilde{\sigma}^4 \in L^1_{\mathrm{loc}}(x) \text{ and } \widetilde{\sigma}^2 = \sigma^2 \ \mu_L\text{-a.e. in a neighborhood of } x\big\}.$$

As a corollary,

$$\begin{aligned} &\widetilde{\mathsf{P}} \ll \mathsf{P} \iff \mathsf{P} \ll \widetilde{\mathsf{P}} \iff \widetilde{\mathsf{P}} = \mathsf{P} \iff \widetilde{b} = 0 \text{ and } \widetilde{\sigma}^2 = \sigma^2 \ \ \mu_L\text{-a.e.}, \\ &\widetilde{\mathsf{P}} \overset{\mathrm{loc}}{\ll} \mathsf{P} \iff \mathsf{P} \overset{\mathrm{loc}}{\ll} \widetilde{\mathsf{P}} \iff \widetilde{b}^2/\widetilde{\sigma}^4 \in L^1_{\mathrm{loc}}(\mathbb{R}) \text{ and } \widetilde{\sigma}^2 = \sigma^2 \ \ \mu_L\text{-a.e.}, \\ &\mathsf{P}_0 \perp \widetilde{\mathsf{P}}_0 \iff \widetilde{b}^2/\widetilde{\sigma}^4 \notin L^1_{\mathrm{loc}}(x_0) \text{ or } \forall \varepsilon > 0,\ \mu_L((x_0 - \varepsilon, x_0 + \varepsilon) \cap \{\widetilde{\sigma}^2 \ne \sigma^2\}) > 0. \end{aligned}$$

Some facts concerning the qualitative behaviour of solutions of SDEs (these are needed in the proofs of results of Section 5) are given in the Appendix.

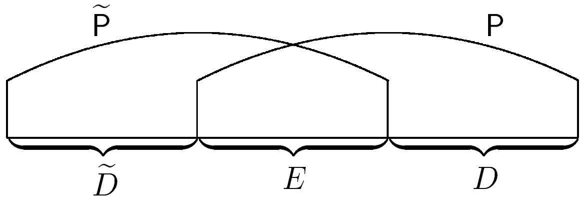
A shortened version of this paper appeared as [5].

2 Separating Times

2.1. Mutual arrangement of a pair of measures on a measurable space. Let P and $\widetilde{\mathsf{P}}$ be probability measures on a measurable space $(\Omega, \mathcal{F})$. The following result is well known.

Proposition 2.1. *There exists a decomposition* $\Omega = E \sqcup D \sqcup \widetilde{D}$, $E, D, \widetilde{D} \in \mathcal{F}$ *such that* $\widetilde{\mathsf{P}} \sim \mathsf{P}$ *on the set* E *and* $\mathsf{P}(\widetilde{D}) = \widetilde{\mathsf{P}}(D) = 0$ (*here* "$\sqcup$" *denotes the disjoint union*). *This decomposition is unique* $\mathsf{P}, \widetilde{\mathsf{P}}$*-a.s.*

Remarks. (i) For the above decomposition, we have $\widetilde{\mathsf{P}} \sim \mathsf{P}$ on E and $\widetilde{\mathsf{P}} \perp \mathsf{P}$ on E^c (here E^c denotes the complement to E). The decomposition $\Omega = E \sqcup E^c$ with these properties is also unique $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.



`Figure 1.` Mutual arrangement of a pair of measures on a measurable space

(ii) The sets $E, D, \widetilde{D}$ from Proposition 2.1 can be obtained as:

$$\widetilde{D} = \left\{\frac{d\mathsf{P}}{d\mathsf{Q}} = 0, \frac{d\widetilde{\mathsf{P}}}{d\mathsf{Q}} > 0\right\}, \quad E = \left\{\frac{d\mathsf{P}}{d\mathsf{Q}} > 0, \frac{d\widetilde{\mathsf{P}}}{d\mathsf{Q}} > 0\right\}, \quad D = \left\{\frac{d\mathsf{P}}{d\mathsf{Q}} > 0, \frac{d\widetilde{\mathsf{P}}}{d\mathsf{Q}} = 0\right\},$$

where $\mathsf{Q} = \frac{\mathsf{P}+\widetilde{\mathsf{P}}}{2}$.

(iii) Proposition 2.1 admits the following statistical interpretation. Suppose that we deal with the problem of distinguishing between two statistical hypotheses P and $\widetilde{\mathsf{P}}$. Unlike the standard setting in statistics, we do not assume that P and $\widetilde{\mathsf{P}}$ are equivalent. Suppose that an experiment is performed, and an elementary outcome ω is obtained. If $\omega \in D$, we can definitely say that the true hypothesis is P; if $\omega \in \widetilde{D}$, we can definitely say that the true hypothesis is $\widetilde{\mathsf{P}}$; if $\omega \in E$, we cannot say for sure what is the true hypothesis.

The result of Proposition 2.1 is illustrated by Figure 1.

2.2. Mutual arrangement of a pair of measures on a filtered space. Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a right-continuous filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$. Recall that the σ-field $\mathcal{F}_\tau$ (τ is a stopping time) is defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \colon A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for any } t \in [0,\infty)\}.$$

(In particular, $\mathcal{F}_\infty = \mathcal{F}$.)

Let P and $\widetilde{\mathsf{P}}$ be probability measures on $\mathcal{F}$. As usually, P_τ (resp., $\widetilde{\mathsf{P}}_\tau$) denotes the restriction of P (resp., $\widetilde{\mathsf{P}}$) to $\mathcal{F}_\tau$.

In what follows, it will be convenient for us to consider the extended positive half-line $[0,\infty] \cup \{\delta\}$, where δ is an additional point. We order $[0,\infty] \cup \{\delta\}$ in the following way: we take the usual order on $[0,\infty]$ and let $\delta > \infty$.

Definition 2.2. An *extended stopping time* is a map $T : \Omega \to [0,\infty] \cup \{\delta\}$ such that $\{T \le t\} \in \mathcal{F}_t$ for any $t \in [0,\infty]$.

The following theorem is an analog of Proposition 2.1 for a filtered space. A similar statement is proved in [20; Lem. 5.2].

Theorem 2.3. **(i)** *There exists an extended stopping time S such that, for any stopping time τ,*

$$\widetilde{\mathsf{P}}_\tau \sim \mathsf{P}_\tau \text{ on the set } \{\tau < S\}, \tag{2.1}$$

$$\widetilde{\mathsf{P}}_\tau \perp \mathsf{P}_\tau \text{ on the set } \{\tau \ge S\}. \tag{2.2}$$

(ii) *If S' is another extended stopping time with these properties, then $S' = S$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.*

Proof. **(i)** Set $\mathsf{Q} = \frac{\mathsf{P}+\widetilde{\mathsf{P}}}{2}$. Let $(Z_t)_{t\in[0,\infty]}$ and $(\widetilde{Z}_t)_{t\in[0,\infty]}$ denote the density processes of P and $\widetilde{\mathsf{P}}$ with respect to Q (we set $Z_\infty = \frac{d\mathsf{P}}{d\mathsf{Q}}$, $\widetilde{Z}_\infty = \frac{d\widetilde{\mathsf{P}}}{d\mathsf{Q}}$). Let $(\overline{\mathcal{F}}_t)$ denote the Q-completion of the filtration $(\mathcal{F}_t)$. Then the $(\overline{\mathcal{F}}_t, \mathsf{Q})$-martingales Z and $\widetilde{Z}$ have the modifications whose all trajectories are càdlàg. The time

$$\overline{S} = \overline{\inf}\{t \in [0,\infty] : Z_t = 0 \text{ or } \widetilde{Z}_t = 0\}$$

("$\overline{\inf}$" is the same as "inf", except that $\overline{\inf}\,\emptyset = \delta$) is an extended $(\overline{\mathcal{F}}_t)$-stopping time. According to [12; Ch. I, Lem. 1.19], there exists an extended $(\mathcal{F}_t)$-stopping time S such that $S = \overline{S}$ Q-a.s. It follows from [12; Ch. III, Lem. 3.6] that $Z_t\widetilde{Z}_t = 0$ on the stochastic interval $[S,\infty]$ Q-a.s. Consequently, for any $(\mathcal{F}_t)$-stopping time τ, we have $Z_\tau\widetilde{Z}_\tau = 0$ Q-a.s. on $\{\tau \ge S\}$. The equality

$$\frac{d\mathsf{P}_\tau}{d\mathsf{Q}_\tau} = \mathsf{E}_\mathsf{Q}\left(\frac{d\mathsf{P}}{d\mathsf{Q}}\,\Big|\,\mathcal{F}_\tau\right) = \mathsf{E}_\mathsf{Q}(Z_\infty \,|\, \mathcal{F}_\tau) = Z_\tau$$

and the analogous equality for $\frac{d\widetilde{\mathsf{P}}_\tau}{d\mathsf{Q}_\tau}$ complete the proof.

(ii) Proposition 2.1 implies that, for any stopping time τ, the sets $\{\tau \ge S\}$ and $\{\tau \ge S'\}$ coincide $\mathsf{P},\widetilde{\mathsf{P}}$-a.s. This yields the desired statement (one needs to consider only the deterministic τ). □

Definition 2.4. A *separating time for* P *and* $\widetilde{\mathsf{P}}$ is an extended stopping time that satisfies (2.1) and (2.2) for all stopping times τ.

Remarks. (i) It is seen from the proof of Theorem 2.3 (ii) that in defining the separating time one may use only the deterministic τ.

(ii) Theorem 2.3 admits the following statistical interpretation (compare with Remark (iii) after Proposition 2.1). Suppose that we deal with the problem of the sequential distinguishing between two statistical hypotheses P and $\widetilde{\mathsf{P}}$. Assume for example that $(\mathcal{F}_t)$ is the natural filtration of an observed process $(X_t)_{t\ge0}$. Suppose that an experiment is performed, and we are observing a path of X. Then, until time S occurs, we cannot say definitely what the true hypothesis is. But after S occurs, we can say definitely what the true hypothesis is (on the set $\{\widetilde{Z}_S = 0\}$, this is P; on the set $\{Z_S = 0\}$, this is $\widetilde{\mathsf{P}}$).

Corollary 2.5. **(i)** *There exists an extended stopping time S such that, for any stopping time τ,*

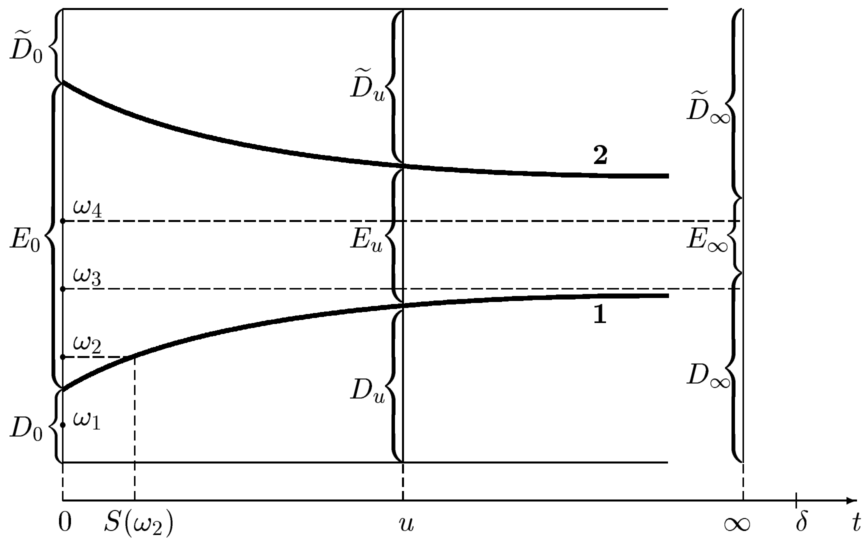
$$\widetilde{\mathsf{P}}_\tau \ll \mathsf{P}_\tau \text{ on the set } \{\tau < S\}, \tag{2.3}$$

$$\widetilde{\mathsf{P}}_\tau \perp \mathsf{P}_\tau \text{ on the set } \{\tau \ge S\}. \tag{2.4}$$

(ii) *If S' is another extended stopping time with these properties, then $S' = S$ $\widetilde{\mathsf{P}}$-a.s.*

Definition 2.6. A *time separating* $\widetilde{\mathsf{P}}$ *from* P is an extended stopping time that satisfies (2.3) and (2.4) for any stopping time τ.

Clearly, a separating time for P and $\widetilde{\mathsf{P}}$ is also a time separating $\widetilde{\mathsf{P}}$ from P. The converse is not true since the former time is unique $\mathsf{P},\widetilde{\mathsf{P}}$-a.s., while the latter time is unique only $\widetilde{\mathsf{P}}$-a.s.



`Figure 2.` Mutual arrangement of a pair of measures on a filtered space (here $S(\omega_1) = 0,\ S(\omega_3) = \infty,\ S(\omega_4) = \delta$)

Informally, Theorem 2.3 states that the measures P and $\widetilde{\mathsf{P}}$ are equivalent up to a random time S and become singular at a time S. The equality $S = \delta$ means that P and $\widetilde{\mathsf{P}}$ never become singular, i.e. they are equivalent up to infinity. Thus, the knowledge of the separating time yields the knowledge of the mutual arrangement of P and $\widetilde{\mathsf{P}}$. This is illustrated by the following result. Its proof is straightforward.

Lemma 2.7. *Let S be a separating time for P and $\widetilde{\mathsf{P}}$. Then*

(i) $\widetilde{\mathsf{P}} \sim \mathsf{P} \iff S = \delta\ \ \mathsf{P}, \widetilde{\mathsf{P}}$-*a.s.*;

(ii) $\widetilde{\mathsf{P}} \ll \mathsf{P} \iff S = \delta\ \ \widetilde{\mathsf{P}}$-*a.s.*;

(iii) $\widetilde{\mathsf{P}} \overset{\text{loc}}{\sim} \mathsf{P} \iff S \geq \infty\ \ \mathsf{P}, \widetilde{\mathsf{P}}$-*a.s.*;

(iv) $\widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P} \iff S \geq \infty\ \ \widetilde{\mathsf{P}}$-*a.s.*;

(v) $\widetilde{\mathsf{P}} \perp \mathsf{P} \iff S \leq \infty\ \ \mathsf{P}, \widetilde{\mathsf{P}}$-*a.s.* $\iff S \leq \infty\ \ \mathsf{P}$-*a.s.*

(vi) $\widetilde{\mathsf{P}}_0 \perp \mathsf{P}_0 \iff S = 0\ \ \mathsf{P}, \widetilde{\mathsf{P}}$-*a.s.* $\iff S = 0\ \ \mathsf{P}$-*a.s.*

`Remark.` Other types of the mutual arrangement of P and $\widetilde{\mathsf{P}}$ are also easily expressed in terms of the separating time. For example, for any $t \in [0, \infty]$,

$$\widetilde{\mathsf{P}}_t \perp \mathsf{P}_t \iff S \leq t\ \ \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s.} \iff S \leq t\ \ \mathsf{P}\text{-a.s.}$$

The mutual arrangement of P and $\widetilde{\mathsf{P}}$ is illustrated by Figure 2. In this figure, the measure $\widetilde{\mathsf{P}}$ "lies above" the curve 1; the measure P "lies below" the curve 2. The decomposition $\Omega = E_t \sqcup D_t \sqcup \widetilde{D}_t$ of Proposition 2.1 for the measurable space $(\Omega, \mathcal{F}_t)$ is obtained by drawing a vertical line corresponding to the time t. Figure 2 shows three decompositions of this type: for $t = 0$, for $t = u \in (0, \infty)$, and for $t = \infty$.

The separating time for P and $\widetilde{\mathsf{P}}$ is illustrated as follows. If $\omega \in D_0 \sqcup \widetilde{D}_0$, then $S(\omega) = 0$ (see $\omega = \omega_1$ in Figure 2). If $\omega \in E_0$, then $S(\omega)$ is the time, at which the horizontal line drawn through the point ω crosses curves 1 or 2 (see $\omega = \omega_2$ in Figure 2). If this line crosses neither curve 1 nor curve 2, then $S = \infty$ in the case $\omega \in D_\infty \sqcup \widetilde{D}_\infty$ (see $\omega = \omega_3$ in Figure 2), and $S = \delta$ in the case $\omega \in E_\infty$ (see $\omega = \omega_4$ in Figure 2).

3 Separating Times for Lévy Processes

Let $D([0,\infty),\mathbb{R}^d)$ denote the space of the càdlàg functions $[0,\infty) \to \mathbb{R}^d$. Let X denote the canonical process on this space, i.e $X_t(\omega) = \omega(t)$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon>0} \sigma(X_s; s \in [0, t+\varepsilon])$ and set $\mathcal{F} = \bigvee_t \mathcal{F}_t$. In what follows, $(\cdot\,,\cdot)$ denotes the scalar product in $\mathbb{R}^d$ and $\|\cdot\|$ denotes the Euclidean norm.

Let P be the distribution of a Lévy process with characteristics (b, c, ν), where $b \in \mathbb{R}^d$, c is a symmetric positively definite $d \times d$ matrix, and ν is a measure on $\mathcal{B}(\mathbb{R}^d)$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (\|x\|^2 \wedge 1)\, \nu(dx) < \infty$. This means that, for any $t \in [0,\infty)$ and $\lambda \in \mathbb{R}^d$,

$$\mathsf{E}_\mathsf{P} e^{i(\lambda, X_t)} = \exp\Big\{ t \Big[i(\lambda, b) - \frac{1}{2}(\lambda, c\lambda) + \int_{\mathbb{R}^d} (e^{i(\lambda,x)} - 1 - i(\lambda,x) I(\|x\| \le 1)) \nu(dx) \Big] \Big\}.$$

(For further information on Lévy processes, see [1], [34], [37; Ch. III, § 1b].) Let $\widetilde{\mathsf{P}}$ be the distribution of a Lévy process with characteristics $(\widetilde{b}, \widetilde{c}, \widetilde{\nu})$.

The following theorem yields an explicit form of the separating time for P and $\widetilde{\mathsf{P}}$. This is actually a reformulation of known results (see, for example, the survey paper [35]) into the language of separating times.

Theorem 3.1. *The separating time S for P and $\widetilde{\mathsf{P}}$ has the following form.*
(i) *If $\mathsf{P} = \widetilde{\mathsf{P}}$, then $S = \delta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.*
(ii) *If $\mathsf{P} \ne \widetilde{\mathsf{P}}$ and*

$$c = \widetilde{c}, \tag{3.1}$$

$$\int_{\mathbb{R}^d} \left(\sqrt{\frac{d\nu}{d(\nu + \widetilde{\nu})}} - \sqrt{\frac{d\widetilde{\nu}}{d(\nu + \widetilde{\nu})}} \right)^2 d(\nu + \widetilde{\nu}) < \infty, \tag{3.2}$$

$$b - \widetilde{b} - \int_{\{\|x\| \le 1\}} x\, d(\nu - \widetilde{\nu}) \in \mathfrak{N}(c), \tag{3.3}$$

where $\mathfrak{N}(c) = \{cx \colon x \in \mathbb{R}^d\}$, then

$$S = \inf\{t \in [0,\infty) \colon \triangle X_t \ne 0, \triangle X_t \notin E\} \ \ \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s.}$$

(we set $\inf \emptyset = \infty$), where $E \in \mathcal{B}(\mathbb{R}^d)$ is a set such that $\widetilde{\nu} \sim \nu$ on E and $\widetilde{\nu} \perp \nu$ on the complement to E.
(iii) *If any of conditions (3.1)–(3.3) is violated, then $S = 0$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.*

`Remarks`. (i) The expression in (3.2) is the Hellinger distance between ν and $\widetilde{\nu}$.
(ii) If (3.2) is true, then $\int_{\{\|x\| \le 1\}} \|x\|\, d\|\nu - \widetilde{\nu}\| < \infty$, where $\|\nu - \widetilde{\nu}\|$ denotes the total variance of the signed measure $\nu - \widetilde{\nu}$ (see [34; Rem. 33.3] or [35; Lem. 2.18]). Thus, the integral in (3.3) is well defined if condition (3.2) is true.

Theorem 3.1, combined with Lemma 2.7, yields the following corollary. This result is known (see [11], [12; Ch. IV, § 4c], [13], [21], [28], [29], [30], [39], [40], [41]).

Corollary 3.2. **(i)** *Either* $\widetilde{\mathsf{P}} = \mathsf{P}$ *or* $\widetilde{\mathsf{P}} \perp \mathsf{P}$.

(ii) *We have* $\widetilde{\mathsf{P}} \stackrel{\text{loc}}{\ll} \mathsf{P}$ *if and only if conditions* (3.1)–(3.3) *and the condition* $\widetilde{\nu} \ll \nu$ *are satisfied.*

(iii) *We have* $\widetilde{\mathsf{P}}_0 \perp \mathsf{P}_0$ *if and only if any of conditions* (3.1)–(3.3) *is violated.*

4 Separating Times for Bessel Processes

Consider the SDE

$$dX_t = \gamma\, dt + 2\sqrt{|X_t|}\, dB_t, \quad X_0 = x_0$$

with $\gamma \geq 0$, $x_0 \geq 0$. It is known that this SDE has a unique solution Q in the sense of Definition 5.2. Moreover, the measure Q is concentrated on positive functions. A process $(Z_t)_{t\in[0,\infty)}$ with the distribution Q is called a *square of a* γ*-dimensional Bessel process started at* $\sqrt{x_0}$. The process $\sqrt{Z}$ is called a γ*-dimensional Bessel process started at* $\sqrt{x_0}$. For more information on Bessel processes, see [2], [3], [6], [32], [33; Ch. XI].

Let X denote the canonical process on $C([0,\infty))$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon>0} \sigma(X_s;\ s \in [0, t+\varepsilon])$ and set $\mathcal{F} = \bigvee_t \mathcal{F}_t$.

Theorem 4.1. *Let* P (*resp.,* $\widetilde{\mathsf{P}}$) *be the distribution of a* γ*-dimensional* (*resp.,* $\widetilde{\gamma}$*-dimensional*) *Bessel process started at* x_0. *Then the separating time* S *for* P *and* $\widetilde{\mathsf{P}}$ *has the following form.*

(i) *If* $\mathsf{P} = \widetilde{\mathsf{P}}$, *then* $S = \delta$ $\mathsf{P}, \widetilde{\mathsf{P}}$*-a.s.*

(ii) *If* $\mathsf{P} \neq \widetilde{\mathsf{P}}$, *then*

$$S = \inf\{t \in [0,\infty) \colon X_t = 0\} \quad \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s.}$$

(*we set* $\inf \emptyset = \infty$).

`Proof.` We should prove only (ii). Set $T_0 = \inf\{t \in [0,\infty) \colon X_t = 0\}$. It follows from [2; Th. 4.1] and the strong Markov property of Bessel processes that $S \leq T_0$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.

Let us prove that $S \geq T_0$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. For $x_0 = 0$, this is obvious, so we assume that $x_0 > 0$. Fix $\varepsilon \in (0, x_0/2)$ and consider the stopping time $T_\varepsilon = \inf\{t \in [0,\infty) : X_t = \varepsilon\}$. Define the map $F_\varepsilon : C([0,\infty)) \to C([0,\infty))$ by $F_\varepsilon(\omega)(t) = \omega(t \wedge T_\varepsilon(\omega))$ and let P^ε denote the image of P under this map. Using Itô's formula, one can check that P^ε is a solution of the SDE

$$dX_t = \frac{\gamma - 1}{2X_t} I(t \leq T_\varepsilon)\, dt + I(t \leq T_\varepsilon)\, dB_t, \quad X_0 = x_0.$$

Let $(\Omega', \mathcal{F}', \mathsf{P}')$ be a probability space with a Brownian motion $(W_t)_{t\in[0,\infty)}$. Consider the space $(C([0,\infty)) \times \Omega', \mathcal{F} \times \mathcal{F}', \mathsf{P}^\varepsilon \times \mathsf{P}')$ and let Q^ε be the distribution of the process

$$Z_t = X_t + \int_0^t I(s > T_\varepsilon) dW_s, \quad t \in [0,\infty).$$

Then Q^ε is a solution of the SDE

$$dX_t = \frac{\gamma - 1}{2X_t} I(t \leq T_\varepsilon)\, dt + dB_t, \quad X_0 = x_0.$$

Similarly, using the measure $\widetilde{\mathsf{P}}$, we define the measure $\widetilde{\mathsf{Q}}^\varepsilon$ that is a solution of the SDE

$$dX_t = \frac{\widetilde{\gamma}-1}{2X_t} I(t \le T_\varepsilon)\,dt + dB_t, \quad X_0 = x_0.$$

Since the drift coefficients $\frac{\gamma-1}{2X_t} I(t \le T_\varepsilon)$ and $\frac{\widetilde{\gamma}-1}{2X_t} I(t \le T_\varepsilon)$ are bounded, we get by Girsanov's theorem that $\widetilde{\mathsf{Q}}^\varepsilon \overset{\text{loc}}{\sim} \mathsf{Q}^\varepsilon$. The obvious equalities $\mathsf{P}^\varepsilon = \mathsf{Q}^\varepsilon \circ F_\varepsilon^{-1}$ and $\widetilde{\mathsf{P}}^\varepsilon = \widetilde{\mathsf{Q}}^\varepsilon \circ F_\varepsilon^{-1}$ yield that $\widetilde{\mathsf{P}}^\varepsilon \overset{\text{loc}}{\sim} \mathsf{P}^\varepsilon$. One can verify that $\mathsf{P}^\varepsilon|\mathcal{F}_{T_{2\varepsilon}} = \mathsf{P}|\mathcal{F}_{T_{2\varepsilon}}$ and $\widetilde{\mathsf{P}}^\varepsilon|\mathcal{F}_{T_{2\varepsilon}} = \widetilde{\mathsf{P}}|\mathcal{F}_{T_{2\varepsilon}}$. Consequently, $\widetilde{\mathsf{P}}|\mathcal{F}_{t\wedge T_{2\varepsilon}} \sim \mathsf{P}|\mathcal{F}_{t\wedge T_{2\varepsilon}}$ for any $t \in [0,\infty)$. Since $t \in [0,\infty)$ and $\varepsilon \in (0, x_0/2)$ are arbitrary, we get the desired inequality $S \ge T_0$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. The proof is completed. □

It is known that if $0 \le \gamma < 2$, then a γ-dimensional Bessel process started at a strictly positive point hits zero with probability one; if $\gamma \ge 2$, then a γ-dimensional Bessel process started at a strictly positive point never hits zero with probability one. Theorem 4.1, combined with Lemma 2.7 and with these properties, yields

Corollary 4.2. **(i)** *Either* $\widetilde{\mathsf{P}} = \mathsf{P}$ *or* $\widetilde{\mathsf{P}} \perp \mathsf{P}$.
(ii) *If* $\widetilde{\mathsf{P}} \ne \mathsf{P}$ *and* $x_0 = 0$, *then* $\widetilde{\mathsf{P}}_0 \perp \mathsf{P}_0$.
(iii) *Let* $\widetilde{\mathsf{P}} \ne \mathsf{P}$ *and* $x_0 > 0$. *Then* $\widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P} \Longleftrightarrow \widetilde{\gamma} \ge 2$.

This corollary generalizes the result of [2; Th. 4.1].

5 Separating Times for Solutions of SDEs

5.1. Basic definitions. We consider one-dimensional SDEs of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \quad X_0 = x_0, \tag{5.1}$$

where b and σ are Borel functions $\mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$.

The standard definition of a solution, which goes back to I.V. Girsanov [9], is as follows.

Definition 5.1. A *solution* of (5.1) is a pair (Y, B) of continuous adapted processes on a filtered probability space $\big(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,\infty)}, \mathsf{Q}\big)$ such that

i) B is a $(\mathcal{G}_t, \mathsf{Q})$-Brownian motion;

ii) for any $t \in [0,\infty)$,

$$\int_0^t (|b(Y_s)| + \sigma^2(Y_s))\,ds < \infty \quad \mathsf{Q}\text{-a.s.};$$

iii) for any $t \in [0,\infty)$,

$$Y_t = x_0 + \int_0^t b(Y_s)\,ds + \int_0^t \sigma(Y_s)\,dB_s \quad \mathsf{Q}\text{-a.s.}$$

`Remark`. A solution in the sense of Definition 5.1 is sometimes called a *weak solution.*

In what follows, it will be convenient for us to treat a solution as a solution of the corresponding martingale problem, i.e. as a measure on the space $C([0,\infty))$ of continuous functions. The corresponding definition goes back to D.W. Stroock and S.R.S. Varadhan [43]. Let X denote the canonical process on $C([0,\infty))$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon>0} \sigma(X_s;\ s \in [0, t+\varepsilon])$ and set $\mathcal{F} = \bigvee_t \mathcal{F}_t$.

Definition 5.2. A *solution* of (5.1) is a probability measure P on $\mathcal{F}$ such that

i) $\mathsf{P}(X_0 = x_0) = 1$;

ii) for any $t \in [0,\infty)$,

$$\int_0^t (|b(X_s)| + \sigma^2(X_s))\,ds < \infty \quad \mathsf{P}\text{-a.s.};$$

iii) the process

$$M_t = X_t - \int_0^t b(X_s)\,ds, \quad t \in [0,\infty)$$

is an $(\mathcal{F}_t, \mathsf{P})$-local martingale with the quadratic variation

$$\langle M\rangle_t = \int_0^t \sigma^2(X_s)\,ds, \quad t \in [0,\infty).$$

The following statement (see, for example, [19; § 5.4.B]) shows the relationship between Definitions 5.1 and 5.2.

Proposition 5.3. **(i)** *Let* (Y, B) *be a solution of* (5.1) *in the sense of Definition* 5.1. *Set* $\mathsf{P} = \mathrm{Law}(Y_t;\ t \in [0,\infty))$. *Then* P *is a solution of* (5.1) *in the sense of Definition* 5.2.

(ii) *Let* P *be a solution of* (5.1) *in the sense of Definition* 5.2. *Then there exist a filtered probability space* $\big(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,\infty)}, \mathsf{Q}\big)$ *and a pair of processes* (Y, B) *on this space such that* (Y, B) *is a solution of* (5.1) *in the sense of Definition* 5.1 *and* $\mathrm{Law}(Y_t;\ t \in [0,\infty)) = \mathsf{P}$.

5.2. Exploding solutions. Definitions 5.1 and 5.2 do not include exploding solutions. However, we need to consider them. Let us introduce some notations.

Let us add a point Δ to the real line and let $C_\Delta([0,\infty))$ denote the space of functions $f : [0,\infty) \to \mathbb{R} \cup \{\Delta\}$ with the property: there exists a time $\zeta(f) \in [0,\infty]$ such that f is continuous on $[0, \zeta(f))$, $f = \Delta$ on $[\zeta(f), \infty))$, and if $0 < \zeta(f) < \infty$, then $\lim_{t\uparrow\zeta(f)} f(t) = \infty$ or $\lim_{t\uparrow\zeta(f)} f(t) = -\infty$. The time $\zeta(f)$ is called the *explosion time of* f. Below in this subsection, X denotes the canonical process on $C_\Delta([0,\infty))$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon>0} \sigma(X_s;\ s \in [0, t+\varepsilon])$ and set $\mathcal{F} = \bigvee_t \mathcal{F}_t$. Let ζ denote the explosion time of the process X.

The next definition is a generalization of Definition 5.2 for the case of exploding solutions.

Definition 5.4. A *solution* of (5.1) is a probability measure P on $\mathcal{F}$ such that

i) $\mathsf{P}(X_0 = x_0) = 1$;

ii) for any $t \in [0,\infty)$ and $n \in \mathbb{N}$ such that $n > |x_0|$,

$$\int_0^{t\wedge\tau_n} (|b(X_s)| + \sigma^2(X_s))\,ds < \infty \quad \mathsf{P}\text{-a.s.},$$

where $\tau_n = \inf\{t \in [0,\infty) \colon |X_t| = n\}$ (we set $\inf\emptyset = \infty$);

iii) for any $n \in \mathbb{N}$ such that $n > |x_0|$, the process

$$M_t^n = X_{t\wedge\tau_n} - \int_0^{t\wedge\tau_n} b(X_s)\,ds, \quad t \in [0,\infty)$$

is an $(\mathcal{F}_t, \mathsf{P})$-local martingale with the quadratic variation

$$\langle M^n\rangle_t = \int_0^{t\wedge\tau_n} \sigma^2(X_s)\,ds, \quad t \in [0,\infty).$$

Clearly, if P is a solution of (5.1) in the sense of Definition 5.4 and $\zeta = \infty$ P-a.s., then the restriction of P to $C([0,\infty))$ is a solution of (5.1) in the sense of Definition 5.2. Conversely, if P is a solution of (5.1) in the sense of Definition 5.2, then there exists a unique extension of the measure P to $C_\Delta([0,\infty))$ that is a solution of (5.1) in the sense of Definition 5.4.

Definition 5.5. A Borel function $f:\mathbb{R}\to[0,\infty)$ is *locally integrable at a point* $a\in[-\infty,\infty]$ if there exists a neighborhood U of a such that $\int_U f(x)\,dx<\infty$. (A neighborhood of ∞ is a ray of the form (x,∞); a neighborhood of $-\infty$ is a ray of the form $(-\infty,x)$.) Notation: $f\in L^1_{\mathrm{loc}}(a)$.

A function f is *locally integrable on a set* $A\subseteq[-\infty,\infty]$ if f is locally integrable at each point of this set. Notation: $f\in L^1_{\mathrm{loc}}(A)$.

Below we shall use the following result (see [7]).

Proposition 5.6 (Engelbert, Schmidt). *Suppose that the coefficients b and σ of* (5.1) *satisfy the conditions:*

$$\sigma(x)\neq 0\ \forall x\in\mathbb{R}, \tag{5.2}$$

$$\frac{1+|b|}{\sigma^2}\in L^1_{\mathrm{loc}}(\mathbb{R}). \tag{5.3}$$

Then, for any starting point $x_0\in\mathbb{R}$, there exists a unique solution of (5.1) *in the sense of Definition* 5.4.

For the information on the qualitative behaviour of the solution of (5.1) under conditions (5.2) and (5.3), see the Appendix.

5.3. Explicit form of the separating time. Here we use the notations $\mathcal{F}$, $\mathcal{F}_t$, X, and ζ introduced in Subsection 5.2.

Consider the SDEs

$$dX_t = b(X_t)\,dt+\sigma(X_t)\,dB_t,\quad X_0=x_0, \tag{5.4}$$

$$dX_t = \widetilde{b}(X_t)\,dt+\widetilde{\sigma}(X_t)\,dB_t,\quad X_0=x_0 \tag{5.5}$$

with the same starting point x_0. Let us assume that conditions (5.2), (5.3) and the similar conditions for $\widetilde{b}$, $\widetilde{\sigma}$ are satisfied.

Set

$$\rho(x)=\exp\left\{-\int_0^x \frac{2b(y)}{\sigma^2(y)}\,dy\right\},\quad x\in\mathbb{R}, \tag{5.6}$$

$$s(x)=\int_0^x \rho(y)\,dy,\quad x\in\mathbb{R}, \tag{5.7}$$

$$s(\infty)=\lim_{x\to\infty}s(x), \tag{5.8}$$

$$s(-\infty)=\lim_{x\to-\infty}s(x). \tag{5.9}$$

Similarly, we define $\widetilde{\rho}$, $\widetilde{s}$, $\widetilde{s}(\infty)$, and $\widetilde{s}(-\infty)$ through $\widetilde{b}$ and $\widetilde{\sigma}$. Let μ_L denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

We say that a point $x \in \mathbb{R}$ is *good* if there exists a neighborhood U of x such that $\sigma^2 = \widetilde{\sigma}^2$ μ_L-a.e. on U and $(b-\widetilde{b})^2/\sigma^4 \in L^1_{\text{loc}}(x)$. We say that the point ∞ is *good* if all the points from $[x_0, \infty)$ are good and

$$s(\infty) < \infty, \tag{5.10}$$

$$(s(\infty) - s)\frac{(b-\widetilde{b})^2}{\rho\sigma^4} \in L^1_{\text{loc}}(\infty). \tag{5.11}$$

We say that the point $-\infty$ is *good* if all the points from $(-\infty, x_0]$ are good and

$$s(-\infty) > -\infty, \tag{5.12}$$

$$(s - s(-\infty))\frac{(b-\widetilde{b})^2}{\rho\sigma^4} \in L^1_{\text{loc}}(-\infty). \tag{5.13}$$

Let A denote the complement to the set of good points in $[-\infty, \infty]$. Clearly, A is closed in $[-\infty, \infty]$. Let us define

$$A^\varepsilon = \{x \in [-\infty, \infty] \colon \rho(x, A) < \varepsilon\},$$

where $\rho(x, y) = |\operatorname{arctg} x - \operatorname{arctg} y|$, $x, y \in [-\infty, \infty]$ (we set $\emptyset^\varepsilon = \emptyset$).

The main result of this section is the following theorem. Its proof is given in Subsection 5.5.

Theorem 5.7. *Suppose that b, σ, $\widetilde{b}$, $\widetilde{\sigma}$ satisfy conditions* (5.2) *and* (5.3). *Let* P *and* $\widetilde{\mathsf{P}}$ *denote the solutions of* (5.4) *and* (5.5) *in the sense of Definition* 5.4. *Then the separating time S for* P *and* $\widetilde{\mathsf{P}}$ *has the following form.*

(i) *If* $\mathsf{P} = \widetilde{\mathsf{P}}$, *then* $S = \delta$ $\mathsf{P}, \widetilde{\mathsf{P}}$*-a.s.*

(ii) *If* $\mathsf{P} \neq \widetilde{\mathsf{P}}$, *then*

$$S = \sup_n \overline{\inf}\{t \in [0, \infty) \colon X_t \in A^{1/n}\} \;\; \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s.},$$

where "$\overline{\inf}$*" is the same as "*$\inf$*", except that* $\overline{\inf}\, \emptyset = \delta$.

`Remarks.` (i) Let us explain the structure of S in case (ii). Denote by α the "bad point that is closest to x_0 from the left side", i.e.

$$\alpha = \begin{cases} \sup\{x : x \in [-\infty, x_0] \cap A\} & \text{if } [-\infty, x_0] \cap A \neq \emptyset, \\ \Delta & \text{if } [-\infty, x_0] \cap A = \emptyset. \end{cases} \tag{5.14}$$

Let us consider the "hitting time of α":

$$U = \begin{cases} \delta & \text{if } \alpha = \Delta, \\ \delta & \text{if } \alpha = -\infty \text{ and } \varliminf_{t \uparrow \zeta} X_t > -\infty, \\ \zeta & \text{if } \alpha = -\infty \text{ and } \varliminf_{t \uparrow \zeta} X_t = -\infty, \\ \overline{T}_\alpha & \text{if } \alpha > -\infty, \end{cases}$$

where $\overline{T}_\alpha = \overline{\inf}\{t \in [0,\infty) \colon X_t = \alpha\}$. Similarly, denote by γ the "bad point that is closest to x_0 from the right side" and denote by V the "hitting time of γ". Then $S = U \wedge V$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. (This follows from Proposition A.1.)

(ii) Suppose that $[x_0,\infty) \subseteq [-\infty,\infty] \setminus A$. Combining Theorem 5.7 with results of Appendix, we get that the pair of conditions (5.10), (5.11) is equivalent to the inequality $\mathsf{P}(\{S = \delta\} \cap (B_+ \cup C_+)) > 0$, where B_+ and C_+ are defined in the Appendix. By the definition of a separating time, the latter condition is equivalent to the inequality $\widetilde{\mathsf{P}}(\{S = \delta\} \cap (B_+ \cup C_+)) > 0$. Applying once more Theorem 5.7 (to the measures $\widetilde{\mathsf{P}}$ and P rather than P and $\widetilde{\mathsf{P}}$) and results of Appendix, we get that this condition is, in turn, equivalent to the pair

$$\widetilde{s}(\infty) < \infty, \tag{5.15}$$

$$(\widetilde{s}(\infty) - \widetilde{s})\frac{(b-\widetilde{b})^2}{\widetilde{\rho}\,\widetilde{\sigma}^4} \in L^1_{\mathrm{loc}}(\infty). \tag{5.16}$$

Thus, assuming that $[x_0,\infty) \subseteq [-\infty,\infty]\setminus A$, we get the equivalence between (5.10)+(5.11) and (5.15)+(5.16). A similar remark is true for (5.12)+(5.13).

Theorem 5.7, combined with Lemma 2.7 and Propositions A.1–A.3, yields several corollaries concerning the mutual arrangement of P and $\widetilde{\mathsf{P}}$. In order to formulate them, let us introduce the conditions:

$$\widetilde{s}(\infty) = \infty, \tag{5.17}$$

$$\widetilde{s}(\infty) < \infty \ \text{ and } \ \frac{\widetilde{s}(\infty) - \widetilde{s}}{\widetilde{\rho}\,\widetilde{\sigma}^2} \notin L^1_{\mathrm{loc}}(\infty), \tag{5.18}$$

$$\widetilde{s}(\infty) < \infty \ \text{ and } \ (\widetilde{s}(\infty) - \widetilde{s})\frac{(b-\widetilde{b})^2}{\widetilde{\rho}\,\widetilde{\sigma}^4} \in L^1_{\mathrm{loc}}(\infty). \tag{5.19}$$

Condition (5.17) means that the paths of the canonical process X under the measure $\widetilde{\mathsf{P}}$ do not tend to ∞ as $t \to \infty$ (see Proposition A.2). Condition (5.18) means that the paths of the canonical process X with a strictly positive $\widetilde{\mathsf{P}}$-probability tend to ∞ as $t \to \infty$, but do not explode into ∞, i.e. the explosion time for them is ∞ (see Proposition A.2). Condition (5.19) is the pair (5.15), (5.16). Similarly, we introduce the conditions at $-\infty$:

$$\widetilde{s}(-\infty) = -\infty, \tag{5.20}$$

$$\widetilde{s}(-\infty) > -\infty \ \text{ and } \ \frac{\widetilde{s} - \widetilde{s}(-\infty)}{\widetilde{\rho}\,\widetilde{\sigma}^2} \notin L^1_{\mathrm{loc}}(-\infty), \tag{5.21}$$

$$\widetilde{s}(-\infty) > -\infty \ \text{ and } \ (\widetilde{s} - \widetilde{s}(-\infty))\frac{(b-\widetilde{b})^2}{\widetilde{\rho}\,\widetilde{\sigma}^4} \in L^1_{\mathrm{loc}}(-\infty). \tag{5.22}$$

Corollary 5.8. *Under the assumptions of Theorem* 5.7, *we have* $\widetilde{\mathsf{P}} \ll \mathsf{P}$ *if and only if at least one of conditions* (a)–(d) *below is satisfied:*

(a) $\mathsf{P} = \widetilde{\mathsf{P}}$;

(b) (5.17), (5.22), *and* (5.23) *are satisfied;*

(c) (5.19), (5.20), *and* (5.23) *are satisfied;*

(d) (5.19), (5.22), *and* (5.23) *are satisfied.*

Corollary 5.9. *Under the assumptions of Theorem* 5.7*, we have* $\widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P}$ *if and only if the condition*

$$\sigma^2 = \widetilde{\sigma}^2 \ \mu_L\text{-a.e. and } \frac{(b-\widetilde{b})^2}{\sigma^4} \in L^1_{\text{loc}}(\mathbb{R}), \tag{5.23}$$

at least one of conditions (5.17)–(5.19)*, and at least one of conditions* (5.20)–(5.22) *are satisfied.*

Remark. The result of Corollary 5.9 is closely connected with the result of Orey [31], where a criterion for the local absolute continuity of regular continuous strong Markov families is provided.

Corollary 5.10. *Under the assumptions of Theorem* 5.7*, we have* $\widetilde{\mathsf{P}} \perp \mathsf{P}$ *if and only if* $\widetilde{\mathsf{P}} \neq \mathsf{P}$ *and* $-\infty, \infty \in A$*.*

Corollary 5.11. *Under the assumptions of Theorem* 5.7*, we have* $\widetilde{\mathsf{P}}_0 \perp \mathsf{P}_0$ *if and only if* $x_0 \in A$*.*

5.4. Examples. In this subsection, we give 9 examples, which show various types of the mutual arrangement of P and $\widetilde{\mathsf{P}}$ from the point of view of their (local) absolute continuity, and singularity. The proofs are straightforward applications of Theorem 5.7 (it is convenient to use also Remark (ii) following Theorem 5.7). One should also take into account the results on the qualitative behaviour of solutions of SDEs that are described in Appendix. In particular, these results imply that a solution P of SDE (5.1) satisfying condition (5.3) with $\sigma \equiv 1$, has the following properties:

- If b is a constant in the neighborhood of $+\infty$, then $\mathsf{P}(\{\zeta < \infty, \lim_{t\uparrow\zeta} X_t = +\infty) = 0$.
- If b is a strictly positive constant in the neighborhood of $+\infty$, then $\mathsf{P}(\lim_{t\to\infty} X_t = +\infty) > 0$.
- If moreover b is positive in the neighborhood of $-\infty$, then $\mathsf{P}(\lim_{t\to\infty} X_t = +\infty) = 1$.
- If $b(x) = x^2$ in the neighborhood of $+\infty$, then $\mathsf{P}(\zeta < \infty, \lim_{t\uparrow\zeta} X_t = +\infty) > 0$.
- If moreover b is positive in the neighborhood of $-\infty$, then $\mathsf{P}(\zeta < \infty, \lim_{t\to\infty} X_t = +\infty) = 1$.

In all the examples below, $\sigma = \widetilde{\sigma} \equiv 1$, $x_0 = 0$, and we specify only b and $\widetilde{b}$.

We use the notation $\widetilde{\mathsf{P}} \triangle \mathsf{P}$ to denote that P and $\widetilde{\mathsf{P}}$ are in a general position, i.e. $\widetilde{\mathsf{P}} \not\ll \mathsf{P}$, $\mathsf{P} \not\ll \widetilde{\mathsf{P}}$, $\widetilde{\mathsf{P}} \not\perp \mathsf{P}$.

Example 5.12. *If*

$$b \equiv 1, \quad \widetilde{b}(x) = 1 + I(0 < x < 1),$$

then

$$\widetilde{\mathsf{P}} \neq \mathsf{P}, \quad \widetilde{\mathsf{P}} \sim \mathsf{P}.$$

Example 5.13. *If*

$$b(x) = I(x > 0) - I(x < 0), \quad \widetilde{b} \equiv 1,$$

then

$$\widetilde{\mathsf{P}} \ll \mathsf{P}, \quad \mathsf{P} \not\ll \widetilde{\mathsf{P}}, \quad \mathsf{P} \overset{\text{loc}}{\ll} \widetilde{\mathsf{P}}.$$

Example 5.14. *If*

$$b(x) = I(x > 0) - x^2 I(x < 0), \quad \widetilde{b} \equiv 1,$$

then

$$\widetilde{\mathsf{P}} \ll \mathsf{P}, \quad \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}.$$

Example 5.15. *If*

$$b(x) = I(x > 0) - I(x < 0), \quad \widetilde{b}(x) = I(x > 0) - 2I(x < 0),$$

then

$$\widetilde{\mathsf{P}} \triangle \mathsf{P}, \quad \widetilde{\mathsf{P}} \overset{\text{loc}}{\sim} \mathsf{P}.$$

Example 5.16. *If*

$$b(x) = I(x > 0) - x^2 I(x < 0), \quad \widetilde{b}(x) = I(x > 0) - I(x < 0),$$

then

$$\widetilde{\mathsf{P}} \triangle \mathsf{P}, \quad \widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P}, \quad \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}.$$

Example 5.17. *If*

$$b \equiv 1, \quad \widetilde{b}(x) = 1 + \frac{I(-1 < x < 0)}{\sqrt{x+1}},$$

then

$$\widetilde{\mathsf{P}} \triangle \mathsf{P}, \quad \widetilde{\mathsf{P}} \overset{\text{loc}}{\not\ll} \mathsf{P}, \quad \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}.$$

Example 5.18. *If*

$$b \equiv 0, \quad \widetilde{b} \equiv 1,$$

then

$$\widetilde{\mathsf{P}} \perp \mathsf{P}, \quad \widetilde{\mathsf{P}} \overset{\text{loc}}{\sim} \mathsf{P}.$$

Example 5.19. *If*

$$b(x) = x^2, \quad \widetilde{b} \equiv 0,$$

then

$$\widetilde{\mathsf{P}} \perp \mathsf{P}, \quad \widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P}, \quad \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}.$$

Example 5.20. *If*

$$b \equiv 0, \quad \widetilde{b}(x) = \frac{I(0 < x < 1)}{\sqrt{x}},$$

then

$$\widetilde{\mathsf{P}} \perp \mathsf{P}, \quad \widetilde{\mathsf{P}} \overset{\text{loc}}{\not\ll} \mathsf{P}, \quad \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}.$$

Type of arrangement	Example
$\widetilde{\mathsf{P}} = \mathsf{P}$	trivial
$\widetilde{\mathsf{P}} \neq \mathsf{P},\ \widetilde{\mathsf{P}} \sim \mathsf{P}$	Example 5.12
$\widetilde{\mathsf{P}} \ll \mathsf{P},\ \mathsf{P} \not\ll \widetilde{\mathsf{P}},\ \mathsf{P} \overset{\text{loc}}{\ll} \widetilde{\mathsf{P}}$	Example 5.13
$\widetilde{\mathsf{P}} \ll \mathsf{P},\ \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}$	Example 5.14
$\widetilde{\mathsf{P}} \triangle \mathsf{P},\ \widetilde{\mathsf{P}} \overset{\text{loc}}{\sim} \mathsf{P}$	Example 5.15
$\widetilde{\mathsf{P}} \triangle \mathsf{P},\ \widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P},\ \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}$	Example 5.16
$\widetilde{\mathsf{P}} \triangle \mathsf{P},\ \widetilde{\mathsf{P}} \overset{\text{loc}}{\not\ll} \mathsf{P},\ \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}$	Example 5.17
$\mathsf{P} \perp \mathsf{P},\ \widetilde{\mathsf{P}} \overset{\text{loc}}{\sim} \mathsf{P}$	Example 5.18
$\widetilde{\mathsf{P}} \perp \mathsf{P},\ \widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P},\ \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}$	Example 5.19
$\widetilde{\mathsf{P}} \perp \mathsf{P},\ \widetilde{\mathsf{P}} \overset{\text{loc}}{\not\ll} \mathsf{P},\ \mathsf{P} \overset{\text{loc}}{\not\ll} \widetilde{\mathsf{P}}$	Example 5.20

Table 1. Various possible types of the mutual arrangement of P and $\widetilde{\mathsf{P}}$ (up to the symmetry between P and $\widetilde{\mathsf{P}}$)

Examples 5.12–5.20 show that all the possible types of the mutual arrangement of P and $\widetilde{\mathsf{P}}$ can be realized. However, the lemma below shows that the types of the mutual arrangement that appear in Examples 5.14, 5.16, and 5.19 can be realized only if P explodes. (In Examples 5.12, 5.13, 5.15, 5.17, 5.18, and 5.20 the measures P and $\widetilde{\mathsf{P}}$ do not explode.)

Lemma 5.21. *Suppose that P does not explode and $\widetilde{\mathsf{P}} \overset{\text{loc}}{\ll} \mathsf{P}$. Then $\mathsf{P} \overset{\text{loc}}{\ll} \widetilde{\mathsf{P}}$.*

Proof. Let S be the separating time for P and $\widetilde{\mathsf{P}}$. By Lemma 2.7, $\widetilde{\mathsf{P}}(S \geq \infty) = 1$. It follows from Theorem 5.7, combined with Proposition A.3 (i), that all the points of $(-\infty, \infty)$ are good. As P does not explode, $\mathsf{P}(S \geq \infty) = 1$. One more application of Lemma 2.7 yields $\mathsf{P} \overset{\text{loc}}{\ll} \widetilde{\mathsf{P}}$. □

Remark. Example 5.19 reveals an interesting effect. Suppose that we are observing a path of the process X and are trying to distinguish between the hypotheses P and $\widetilde{\mathsf{P}}$ (given by Example 5.19). If P is the true hypothesis, we will find this out within a finite time of observations. However, if $\widetilde{\mathsf{P}}$ is the true hypothesis, we will find this out only within the infinite time of observations.

5.5. Proof of Theorem 5.7. In the proof of this theorem, we use the techniques of random time-changes and local times. These can be found in [33; Ch. V, § 1; Ch. VI, §§ 1,2]. Below we deal with the following two settings.

Setting 1. Let X denote the canonical process on $C([0,\infty))$. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon>0}\sigma(X_s;\, s\in[0,t+\varepsilon])$ and set $\mathcal{F}=\bigvee_{t\in[0,\infty)}\mathcal{F}_t$.

Setting 2. Let X denote the canonical process on $C_\Delta([0,\infty))$ and ζ denote the explosion time of X. Consider the filtration $\mathcal{F}_t = \bigcap_{\varepsilon>0}\sigma(X_s;\, s\in[0,t+\varepsilon])$ and set $\mathcal{F}=\bigvee_{t\in[0,\infty)}\mathcal{F}_t$.

We begin with a series of auxiliary lemmas.

Lemma 5.22. *In Setting* 1 *or in Setting* 2*, consider an* $(\mathcal{F}_t)$*-stopping time* τ*. Let* ω *and* ω' *be such that* $\tau(\omega)=t_0\in[0,\infty)$ *and* $\omega'(s)=\omega(s)$ *on* $[0,t_0+\varepsilon]$ *for some* $\varepsilon>0$*. Then* $\tau(\omega')=t_0$ *and, for any* $A\in\mathcal{F}_\tau$*,* $\omega\in A \Longleftrightarrow \omega'\in A$*.*

This lemma may be proved by the standard technique. For statements with similar proofs, see, for example, [12; Ch. III, Lem. 2.43], [33; Ch. I, Ex. 4.21], [36; Ch. I, § 2, Lem. 13].

Lemma 5.23. *Let* $Y=(Y_t)_{t\in[0,\infty)}$ *be a continuous process on a probability space* $(\Omega,\mathcal{G},\mathsf{Q})$*. Introduce the filtration* $\mathcal{G}_t^Y=\bigcap_{\varepsilon>0}\sigma(Y_s;\, s\in[0,t+\varepsilon])$*. Let* τ *be a* $(\mathcal{G}_t^Y)$*-stopping time. Then there exists an* $(\mathcal{F}_t)$*-stopping time* ρ *such that* $\tau=\rho(Y)$*, where* $(\mathcal{F}_t)$ *denotes the filtration introduced in Setting* 1*.*

This lemma may be proved similarly to [12; Ch. I, Lem. 1.19].

Lemma 5.24. *Assume that the coefficients* b *and* σ *of* (5.1) *satisfy conditions* (5.2) *and* (5.3)*. Let* P *be a solution of* (5.1) *in the sense of Definition* 5.4 (*so, we consider Setting* 2)*. Then* $\mathcal{F}_0$ *is* P*-trivial.*

`Proof.` This is a consequence of the following result (see [43; Th. 6.2] or [18; Th. 18.11]): if for any starting point $x_0\in\mathbb{R}$, there exists a unique solution P_{x_0} of (5.1), then the family $(X_t,\mathcal{F}_t,\mathsf{P}_x;\, t\in[0,\infty), x\in\mathbb{R})$ possesses the strong Markov property. After applying this result one should note that any strong Markov family satisfies the required zero-one law. □

Lemma 5.25. *Assume that the coefficients* b *and* σ *of* (5.1) *satisfy conditions* (5.2) *and* (5.3) *and that the solution is nonexploding. Let* P *be a solution of* (5.1) *in the sense of Definition* 5.2 (*so, we consider Setting* 1)*. Then, for any* $(\mathcal{F}_t)$*-stopping time* ξ *such that* $\xi>0$ P*-a.s., there exists an* $(\mathcal{F}_t)$*-stopping time* ξ' *such that* $0<\xi'<\xi$ P*-a.s.*

`Proof.` **1)** Define the functions ρ and s by formulas (5.6) and (5.7). Consider the process $Y=s(X)$. Due to the Ito-Tanaka formula (see [33; Ch. VI, Th. 1.5]), Y is a continuous $(\mathcal{F}_t,\mathsf{P})$-local martingale with the quadratic variation $\langle Y\rangle_t=\int_0^t \varkappa^2(Y_u)\,du$, where $\varkappa(x)=\rho(s^{-1}(x))\,\sigma(s^{-1}(x))$, $x\in s(\mathbb{R})$. Since $\sigma(x)\neq 0$ for any $x\in\mathbb{R}$, then P-a.s. the trajectories of $\langle Y\rangle$ are continuous and strictly increasing. Denote by $\overline{\mathcal{F}}$ the P-completion of the σ-field $\mathcal{F}$ and by $(\overline{\mathcal{F}}_t)$ the P-completion of the filtration $(\mathcal{F}_t)$. Define an $(\overline{\mathcal{F}}_t)$-time-change

$$\tau_t=\inf\{s\in[0,\infty)\colon \langle Y\rangle_s>t\},\quad t\in[0,\infty). \tag{5.24}$$

Consider an $(\mathcal{F}'_t,\mathsf{P}')$-Brownian motion W' on some stochastic basis $(\Omega',\mathcal{F}',(\mathcal{F}'_t),\mathsf{P}')$ and set

$$\Omega=C([0,\infty))\times\Omega',\quad \mathcal{G}=\overline{\mathcal{F}}\times\mathcal{F}',\quad \mathcal{G}_t=\bigcap_{\varepsilon>0}\overline{\mathcal{F}}_{\tau_{t+\varepsilon}}\times\mathcal{F}'_{t+\varepsilon},\quad \mathsf{Q}=\mathsf{P}\times\mathsf{P}'.$$

Denote by $\overline{\mathcal{G}}$ the Q-completion of the σ-field $\mathcal{G}$ and by $(\overline{\mathcal{G}}_t)$ the Q-completion of the filtration $(\mathcal{G}_t)$. Consider the stochastic basis $(\Omega, \overline{\mathcal{G}}, (\overline{\mathcal{G}}_t), \mathsf{Q})$. All the random variables and the processes defined on $C([0,\infty))$ or on Ω' can be viewed as random variables and processes on Ω. In what follows, we do not explain on which space we consider a random variable or a process if this is clear from the context.

Set

$$W_t = Y_{\tau_t} + W'_t - W'_{t\wedge\langle Y\rangle_\infty}, \quad t \in [0,\infty). \tag{5.25}$$

By the Dambis-Dubins-Schwartz theorem (see [33; Ch. V, Th. 1.6]), the process $W = (W_t)_{t\in[0,\infty)}$ is a $(\overline{\mathcal{G}}_t, \mathsf{Q})$-Brownian motion with the starting point $s(x_0)$.

As P-a.s. the trajectories of $\langle Y\rangle$ are continuous, we have $\langle Y\rangle_{\tau_t} = t$ P-a.s. on $\{t < \langle Y\rangle_\infty\}$, i.e.

$$\int_0^{\tau_t} \varkappa^2(Y_u)\,du = t \quad \text{P-a.s. on } \{t < \langle Y\rangle_\infty\}.$$

As P-a.s. the trajectories of $\langle Y\rangle$ are strictly increasing, then P-a.s. the trajectories of τ are continuous (however, they may explode). By the change of variables in the Stieltjes integral, we get

$$\int_0^{t} \varkappa^2(Y_{\tau_u})\,d\tau_u = t \quad \text{P-a.s. on } \{t < \langle Y\rangle_\infty\},$$

and therefore,

$$\tau_t = \int_0^{t} \varkappa^{-2}(Y_{\tau_u})\,du \quad \text{P-a.s. on } \{t < \langle Y\rangle_\infty\}.$$

Since $\tau_t \to \infty$ P-a.s. as $t \uparrow \langle Y\rangle_\infty$ and $Y_{\tau_t} = W_t$ for $t < \langle Y\rangle_\infty$, we have

$$\tau_t = \int_0^{t} \varkappa^{-2}(W_u)\,du \quad \text{Q-a.s.}, \quad t \in [0,\infty) \tag{5.26}$$

(here we set $\varkappa(x) = 1$ for $x \notin s(\mathbb{R})$).

Consider the filtration $\mathcal{H}_t^W = \bigcap_{\varepsilon>0} \sigma(W_s;\ s \in [0, t+\varepsilon])$ and let $(\overline{\mathcal{H}}_t^W)$ denote its Q-completion. By (5.26), the process τ viewed as a process on Ω is $(\overline{\mathcal{H}}_t^W)$-adapted. Due to (5.24),

$$\langle Y\rangle_t = \inf\{s \in [0,\infty) : \tau_s > t\} \quad \text{P-a.s.}, \quad t \in [0,\infty).$$

Therefore, the process $\langle Y\rangle$ viewed as a process on Ω is an $(\overline{\mathcal{H}}_t^W)$-time-change. Furthermore, (5.25) implies that $Y_t = W_{\langle Y\rangle_t}$ Q-a.s. Since the right-continuous and Q-complete filtration generated by Y viewed as a process on Ω contains the filtration $(\overline{\mathcal{F}}_t \times \{\emptyset, \Omega'\})$, we have

$$\overline{\mathcal{F}}_t \times \{\emptyset, \Omega'\} \subseteq \overline{\mathcal{H}}^W_{\langle Y\rangle_t}. \tag{5.27}$$

The process τ is an $(\overline{\mathcal{F}}_t \times \{\emptyset, \Omega'\})$-time-change. It follows from (5.27) (see also [33; Ch. V, Ex. 1.12]) that

$$\overline{\mathcal{F}}_{\tau_t} \times \{\emptyset, \Omega'\} \subseteq \overline{\mathcal{H}}^W_{t\wedge\langle Y\rangle_\infty} \subseteq \overline{\mathcal{H}}^W_t. \tag{5.28}$$

2) It is easy to verify that $\langle Y\rangle_\xi$ viewed as a random variable on Ω is an $(\overline{\mathcal{F}}_{\tau_t} \times \{\emptyset, \Omega'\})$-stopping time. By (5.28), $\langle Y\rangle_\xi$ is an $(\overline{\mathcal{H}}_t^W)$-stopping time. Since $\xi > 0$ P-a.s., then $\langle Y\rangle_\xi > 0$ Q-a.s. Furthermore, the σ-field $\overline{\mathcal{H}}_0^W$ is Q-trivial; it is also well known that every stopping time on a complete Brownian filtration is predictable. Hence, there exists an $(\overline{\mathcal{H}}_t^W)$-stopping time η such that

$$0 < \eta < \langle Y\rangle_\xi \quad \text{Q-a.s.} \tag{5.29}$$

It is known (see [12; Ch. I, Lem. 1.19]) that every stopping time with respect to a completion of a right-continuous filtration $(\mathcal{K}_t)$ a.s. coincides with a $(\mathcal{K}_t)$-stopping time. Therefore, we can choose η in such a way that it is an $(\mathcal{H}_t^W)$-stopping time. Due to Lemma 5.23, there exists an $(\mathcal{F}_t)$-stopping time ρ such that

$$\eta = \rho(W) \quad \mathsf{Q}\text{-a.s.} \tag{5.30}$$

Now, define the process $V_t = Y_{\tau_t}$, $t \in [0,\infty)$. (Note that $\{\tau_t = \infty\} = \{\langle Y\rangle_\infty \le t\}$ P-a.s. and on the set $\{\langle Y\rangle_\infty < \infty\}$ the process Y_t tends P-a.s. to a finite random variable Y_∞. Hence, the process V is well defined.) Equations (5.29) and (5.30) imply that $\rho(W) < \langle Y\rangle_\infty$ Q-a.s. Since $V = W^{\langle Y\rangle_\infty}$ Q-a.s., then, by Lemma 5.22, $\rho(W) = \rho(V)$ Q-a.s. The random variables $\rho(V)$ and $\langle Y\rangle_\xi$ are defined on $C([0,\infty))$. Hence, we can write

$$0 < \rho(V) < \langle Y\rangle_\xi \quad \mathsf{P}\text{-a.s.} \tag{5.31}$$

Consider the filtration $\mathcal{F}_t^V$ on $C([0,\infty))$ defined by the formula $\mathcal{F}_t^V = \bigcap_{\varepsilon>0} \sigma(V_s;\, s \in [0,t+\varepsilon])$. Since the process V is $\overline{\mathcal{F}}_{\tau_t}$-adapted and the filtration $\overline{\mathcal{F}}_{\tau_t}$ is right-continuous, we have $\mathcal{F}_t^V \subseteq \overline{\mathcal{F}}_{\tau_t}$. Consequently, $\rho(V)$ is an $(\overline{\mathcal{F}}_{\tau_t})$-stopping time. By [33; Ch. V, Ex. 1.12], $\tau_{\rho(V)}$ is an $(\overline{\mathcal{F}}_t)$-stopping time. Due to [12; Ch. I, Lem. 1.19], there exists an $(\mathcal{F}_t)$-stopping time ξ' such that $\xi' = \tau_{\rho(V)}$ P-a.s. Finally, (5.31) implies that $0 < \xi' < \xi$ P-a.s. □

Now, let us introduce some notations. Suppose that $a, c \in [-\infty,\infty]$. In Setting 1 or in Setting 2, define

$$T_a = \inf\{t \in [0,\infty) \colon X_t = a\}, \tag{5.32}$$
$$T_{a,c} = T_a \wedge T_c. \tag{5.33}$$

Note that if $a = -\infty$ or $a = \infty$, then $T_a = \infty$. Similarly, for a process Y, we use the notations

$$T_a(Y) = \inf\{t \in [0,\infty) \colon Y_t = a\}, \tag{5.34}$$
$$T_{a,c}(Y) = T_a(Y) \wedge T_c(Y). \tag{5.35}$$

Below in this section, we use the notations ρ, s, $s(\infty)$, $s(-\infty)$ introduced in (5.6)–(5.9). Let us define the function $\varkappa$ by the formula

$$\varkappa(x) = \rho(s^{-1}(x))\,\sigma(s^{-1}(x)), \quad x \in s(\mathbb{R}). \tag{5.36}$$

We need a more detailed version of the Engelbert-Schmidt theorem than Proposition 5.6 (see [7]).

Proposition 5.26 (Engelbert, Schmidt). *Suppose that the coefficients b and σ of* (5.1) *satisfy conditions* (5.2) *and* (5.3).

(i) *Then, for any starting point $x_0 \in \mathbb{R}$, there exists a unique solution of* (5.1) *in the sense of Definition* 5.4.

(ii) *Let P_{x_0} denote this solution. Consider a stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,\infty)}, \mathsf{Q})$ with a right-continuous and complete filtration. Let B be a*

$(\mathcal{G}_t, \mathsf{Q})$-Brownian motion with the starting point $s(x_0)$. Define the process $(A_t)_{t\in[0,\infty)}$ and the $(\mathcal{G}_t)$-time-change $(\tau_t)_{t\in[0,\infty)}$ by the formulas

$$A_t = \begin{cases} \int_0^t \varkappa^{-2}(B_s)\,ds & \text{if } t < T_{s(-\infty),s(\infty)}(B), \\ \infty & \text{if } t \ge T_{s(-\infty),s(\infty)}(B), \end{cases} \tag{5.37}$$

$$\tau_t = \inf\{s \in [0,\infty)\colon A_s > t\}. \tag{5.38}$$

Then

$$\mathsf{P}_{x_0} = \mathrm{Law}\big(s^{-1}(B_{\tau_t});\, t \in [0,\infty)\big|\mathsf{Q}\big),$$

where we set $s^{-1}(s(\infty)) = s^{-1}(s(-\infty)) = \Delta$.

`Remark.` Propositions A.1 and A.3 may easily be derived from the second part of Proposition 5.26.

Lemma 5.27. *Assume that the coefficients b and σ of* (5.1) *satisfy conditions* (5.2) *and* (5.3)*. Additionally assume that $s(\infty) < \infty$. Denote by P the solution of* (5.1) *in the sense of Definition* 5.4 *(so, we consider Setting* 2*). Let $a < x_0$ and f be a positive Borel function such that $f/\sigma^2 \in L^1_{\mathrm{loc}}([a,\infty))$.*

(i) *If $(s(\infty) - s)f/(\rho\sigma^2) \in L^1_{\mathrm{loc}}(\infty)$, then*

$$\int_0^\zeta f(X_t)\,dt < \infty \quad \mathsf{P}\text{-a.s. on the set } \{T_a = \infty\}$$

(recall that ζ denotes the explosion time of X).

(ii) *If $(s(\infty) - s)f/(\rho\sigma^2) \notin L^1_{\mathrm{loc}}(\infty)$, then*

$$\int_0^\zeta f(X_t)\,dt = \infty \quad \mathsf{P}\text{-a.s. on the set } \{T_a = \infty\}.$$

`Remark.` Due to Proposition A.1, $\lim_{t\uparrow\zeta} X_t = \infty$ P-a.s. on the set $\{T_a = \infty\}$. Therefore, Lemma 5.27 deals, in fact, with the convergence of some integrals on the trajectories that tend to ∞ or explode to ∞. Clearly, this lemma has its analog for the trajectories that tend to $-\infty$ or explode to $-\infty$.

`Proof of Lemma 5.27.` We prove only the first part. The proof of the second one is analogous.

Consider a stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,\infty)}, \mathsf{Q})$ with a right-continuous and complete filtration and let B be a $(\mathcal{G}_t, \mathsf{Q})$-Brownian motion with the starting point $s(x_0)$. Define the process $(A_t)_{t\in[0,\infty)}$ and the $(\mathcal{G}_t)$-time-change $(\tau_t)_{t\in[0,\infty)}$ by formulas (5.37) and (5.38). Set $\xi = A_{T_{s(-\infty),s(\infty)}(B)-}$.

Proposition 5.26 yields that the convergence of the integral $\int_0^\zeta f(X_t)\,dt$ P-a.s. on the set $\{T_a = \infty\}$ is equivalent to the convergence of the integral $\int_0^\xi f(s^{-1}(B_{\tau_t}))\,dt$ Q-a.s. on the set $\{T_{s(\infty)}(B) < T_{s(a)}(B)\}$. By the change of variables in the Stieltjes integral, we get

$$\int_0^\xi f(s^{-1}(B_{\tau_t}))\,dt = \int_0^\xi f(s^{-1}(B_{\tau_t}))\,dA_{\tau_t} = \int_0^{\tau_\xi} f(s^{-1}(B_t))\,dA_t$$
$$= \int_0^{T_{s(-\infty),s(\infty)}(B)} \frac{f}{\rho^2\sigma^2}(s^{-1}(B_t))\,dt.$$

Set

$$g(x)=\frac{f}{\rho^2\sigma^2}(s^{-1}(x)),\quad x\in s(\mathbb{R}).$$

Since $T_{s(-\infty),s(\infty)}(B)=T_{s(\infty)}(B)$ on the set $\{T_{s(\infty)}(B)<T_{s(a)}(B)\}$, then the problem reduces to investigating the convergence of the integral $\int_0^{T_{s(\infty)}(B)}g(B_t)\,dt$ Q-a.s. on the set $\{T_{s(\infty)}(B)<T_{s(a)}(B)\}$.

Since $(s(\infty)-s)f/(\rho\sigma^2)\in L^1_{\text{loc}}(\infty)$, then

$$\exists\,\varepsilon>0\colon\ \int_{s(\infty)-\varepsilon}^{s(\infty)}(s(\infty)-x)g(x)\,dx<\infty.$$

As $f/\sigma^2\in L^1_{\text{loc}}([a,\infty))$, we have $g\in L^1_{\text{loc}}([s(a),s(\infty)))$. Now, we need to use the results of the paper [2], where the convergence of some integrals associated with Bessel processes is investigated. By [2; Th. 2.2],

$$\int_0^{T_{s(a)}(s(\infty)-Y)}g(s(\infty)-Y)\,dt<\infty\quad \mathsf{R}\text{-a.s.},$$

where Y is a three-dimensional Bessel process started at zero and defined on a probability space with a measure R. Set $Z_t=s(\infty)-Y_t$, $t\in[0,\infty)$. Then

$$\int_0^{U_{s(x_0)}(Z)}g(Z_t)\,dt<\infty\quad \mathsf{R}\text{-a.s. on the set }\{U_{s(x_0)}(Z)<T_{s(a)}(Z)\},$$

where we use the notation $U_c(Z)=\sup\{t\in[0,\infty)\colon Z_t=c\}$. Now, the Williams theorem (see [33; Ch. VII, Cor. 4.6]), combined with the last formula, yields

$$\int_0^{T_{s(\infty)}(B)}g(B_t)\,dt<\infty\quad \mathsf{Q}\text{-a.s. on the set }\{T_{s(\infty)}(B)<T_{s(a)}(B)\}.$$

This completes the proof. □

In what follows, μ_L denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

Lemma 5.28. *Assume that the coefficients b and σ of* (5.1) *satisfy conditions* (5.2) *and* (5.3)*. Additionally assume that $s(-\infty)=-\infty$ and $s(\infty)=\infty$. Denote by* P *the solution of* (5.1) *in the sense of Definition* 5.4 (*so, we consider Setting* 2)*. Let f be a positive Borel function such that $\mu_L(f>0)>0$. Then*

$$\int_0^\infty f(X_t)\,dt=\infty\quad \mathsf{P}\text{-a.s.}$$

(*Let us recall that, by Propositions* A.1 *and* A.2, *$\zeta=\infty$* P*-a.s. whenever $s(\infty)=\infty$ and $s(-\infty)=-\infty$.*)

Remark. Lemmas 5.27 and 5.28 complement each other. Indeed, Lemma 5.27 deals with the convergence of some integrals on the trajectories that tend to ∞ (or to $-\infty$), while Lemma 5.28 deals with the convergence of some integrals on the trajectories that are recurrent.

Proof of Lemma 5.28. Using a reasoning similar to that of the previous lemma, we see that we need to prove the equality $\int_0^\infty g(B_t)\,dt=\infty$ Q-a.s., where $g(x)=$

$\frac{f}{\rho^2\sigma^2}(s^{-1}(x))$, $x \in \mathbb{R}$, and B is a Q-Brownian motion defined on some probability space. It is known that local times of a Brownian motion satisfy $L^x_\infty(B) = \infty$ for all $x \in \mathbb{R}$ (see [33; Ch. VI, Cor. 2.4]). By the occupation times formula (see [33; Ch. VI, Cor. 1.6]),

$$\int_0^\infty g(B_t)\,dt = \int_{\mathbb{R}} g(x) L^x_\infty(B)\,dx = \infty \quad \mathsf{Q}\text{-a.s.}$$

The proof is completed. □

Let Y be a continuous semimartingale on some stochastic basis. Below in this section, we use the notation $L^x_t(Y)$ $(t \in [0,\infty),\ x \in \mathbb{R})$ for the local time of a process Y spent at a point a by a time t. We take versions of local times that are càdlàg in x and use the notation $L^{x-}_t(Y) := \lim_{\varepsilon \downarrow 0} L^{x-\varepsilon}_t(Y)$.

Lemma 5.29. *Assume that the coefficients b, σ and $\widetilde{b}$, $\widetilde{\sigma}$ of* (5.4) *and* (5.5) *satisfy conditions* (5.2) *and* (5.3) *and that the solutions are nonexploding. Let P and $\widetilde{\mathsf{P}}$ be the solutions of* (5.4) *and* (5.5) *in the sense of Definition* 5.2 *(so, we consider Setting* 1*). Suppose that the condition*

$$\forall\, \varepsilon > 0,\ \mu_L((x_0 - \varepsilon, x_0 + \varepsilon) \cap \{\sigma^2 \neq \widetilde{\sigma}^2\}) > 0 \tag{5.39}$$

or the condition

$$\frac{(\widetilde{b} - b)^2}{\sigma^4} \notin L^1_{\mathrm{loc}}(x_0) \tag{5.40}$$

is satisfied. Then $\widetilde{\mathsf{P}}_0 \perp \mathsf{P}_0$ (let us recall that P_0 and $\widetilde{\mathsf{P}}_0$ denote the restrictions of P and $\widetilde{\mathsf{P}}$ to the σ-field $\mathcal{F}_0$).

Proof. **1)** Let us first assume that condition (5.39) holds. By the occupation times formula (see [33; Ch. VI, Cor. 1.6]),

$$\begin{aligned}\int_0^t I_{\{\sigma^2 \neq \widetilde{\sigma}^2\}}(X_u)\sigma^2(X_u)\,du &= \int_0^t I_{\{\sigma^2 \neq \widetilde{\sigma}^2\}}(X_u)\,d\langle X\rangle_u \\ &= \int_{\mathbb{R}} I_{\{\sigma^2 \neq \widetilde{\sigma}^2\}}(x) L^x_t(X)\,dx \quad \mathsf{P}\text{-a.s.}\end{aligned}$$

It follows from [4; Th. 2.7] that $L^{x_0}_t(X) > 0$ and $L^{x_0-}_t(X) > 0$ P-a.s. for any $t > 0$. Therefore, for any $t > 0$,

$$\int_0^t I_{\{\sigma^2 \neq \widetilde{\sigma}^2\}}(X_u)\sigma^2(X_u)\,du > 0 \quad \mathsf{P}\text{-a.s.}$$

Hence, for any $t > 0$,

$$\mathsf{P}\left(\exists\, 0 < s \le t\colon \int_0^s \sigma^2(X_u)\,du \neq \int_0^s \widetilde{\sigma}^2(X_u)\,du\right) = 1,$$

and consequently,

$$\mathsf{P}\left(\forall\, t > 0\ \exists\, 0 < s \le t\colon \int_0^s \sigma^2(X_u)\,du \neq \int_0^s \widetilde{\sigma}^2(X_u)\,du\right) = 1. \tag{5.41}$$

Let us recall that P-quadratic variation (resp., $\widetilde{\mathsf{P}}$-quadratic variation) of X at time s equals $\int_0^s \sigma^2(X_u)\,du$ P-a.s. (resp., $\int_0^s \widetilde{\sigma}^2(X_u)\,du$ $\widetilde{\mathsf{P}}$-a.s.). Therefore, for any sequence (Δ_n) of subdivisions of the interval $[0,s]$ whose diameters tend to 0, we have

$$\int_0^s \sigma^2(X_u)\,du = \mathsf{P}\text{-}\lim_{n\to\infty} \sum_{t_i\in\Delta_n} (X_{t_i} - X_{t_{i-1}})^2$$

and

$$\int_0^s \widetilde{\sigma}^2(X_u)\,du = \widetilde{\mathsf{P}}\text{-}\lim_{n\to\infty} \sum_{t_i\in\Delta_n} (X_{t_i} - X_{t_{i-1}})^2.$$

Now, consider all rational times s. By extracting a.s. converging subsequences and using Cantor's diagonal method, we see that (5.41) implies the desired result $\widetilde{\mathsf{P}}_0 \perp \mathsf{P}_0$.

2) Assume now that condition (5.40) holds. Denote by S the separating time for P and $\widetilde{\mathsf{P}}$. Due to Lemma 5.24, the σ-field $\mathcal{F}_0$ is trivial with respect to each of the measures P and $\widetilde{\mathsf{P}}$. Combining this with Lemma 2.7, we obtain that either $S=0$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s. or $S>0$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s. Let us prove that the second variant is not possible.

Suppose, on the contrary, that $S>0$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s. (or, equivalently, $\widetilde{\mathsf{P}}_0 \not\perp \mathsf{P}_0$). By Lemma 5.25, there exist stopping times τ' and τ'' such that $0<\tau'<S$ P-a.s. and $0<\tau''<S$ $\widetilde{\mathsf{P}}$-a.s. Set $\tau=\tau'\wedge\tau''$. Then it follows from our assumption $\widetilde{\mathsf{P}}_0 \not\perp \mathsf{P}_0$ and from the fact that $\mathcal{F}_0$ is both P- and $\widetilde{\mathsf{P}}$-trivial that $0<\tau<S$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s. Hence, $\widetilde{\mathsf{P}}_\tau \sim \mathsf{P}_\tau$.

Consider the càdlàg $(\mathcal{F}_t,\mathsf{P})$-martingale

$$Z_t = \mathsf{E}_{\mathsf{P}}\left(\frac{d\widetilde{\mathsf{P}}_\tau}{d\mathsf{P}_\tau}\middle|\mathcal{F}_t\right), \quad t\in[0,\infty).$$

Notice that Z is a uniformly integrable martingale with a limit $Z_\infty = \frac{d\widetilde{\mathsf{P}}_\tau}{d\mathsf{P}_\tau}$. Since $Z_\infty>0$ P-a.s., the processes Z and Z_- are strictly positive P-a.s. (see [12; Ch. III, Lem. 3.6]). Set

$$L_t = \int_0^t \frac{1}{Z_{u-}}\,dZ_u, \quad t\in[0,\infty).$$

The $(\mathcal{F}_t,\mathsf{P})$-local martingale L is well defined. Clearly, we have $Z = Z_0\,\mathcal{E}(L)$ (i.e. Z is a stochastic exponent of L). Since P is a unique solution of (5.4), any $(\mathcal{F}_t,\mathsf{P})$-local martingale is a stochastic integral with respect to the local martingale Y (see [12; Ch. III, Th. 4.29]), where Y is the continuous martingale part of the $(\mathcal{F}_t,\mathsf{P})$-semimartingale X, i.e.

$$Y_t = X_t - \int_0^t b(X_u)\,du, \quad t\in[0,\infty).$$

In particular, there exists a predictable process β such that

$$\int_0^t \beta_u^2\,d\langle Y\rangle_u < \infty \quad \mathsf{P}\text{-a.s.}, \quad t\in[0,\infty)$$

and

$$L_t = \int_0^t \beta_u\,dY_u \quad \mathsf{P}\text{-a.s.}, \quad t\in[0,\infty).$$

This yields that the process L is continuous.

Consider the measure $\mathsf{Q} = Z_\infty \cdot \mathsf{P}$. Then $\mathsf{Q}_\tau = \widetilde{\mathsf{P}}_\tau$. It follows from Girsanov's theorem for local martingales (see [12; Ch. III, Th. 3.11]) that the process $Y - \langle Y, L\rangle$ is an $(\mathcal{F}_t, \mathsf{Q})$-local martingale. We have

$$\langle Y, L\rangle_t = \int_0^t \beta_u \, d\langle Y\rangle_u = \int_0^t \beta_u \sigma^2(X_u)\, du \quad \mathsf{P}\text{-a.s.}, \quad t \in [0,\infty).$$

For any $t \in [0,\infty)$, set

$$M_t = \begin{cases} X_{t\wedge\tau} - \displaystyle\int_0^{t\wedge\tau} (b(X_u) + \beta_u\sigma^2(X_u))\, du & \text{if } \displaystyle\int_0^{t\wedge\tau} (|b(X_u)| \\ & \qquad +|\beta_u|\sigma^2(X_u))\, du < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

The process M is finite and continuous with respect to P. Hence, it is finite and continuous with respect to Q. Since $\mathsf{Q}_\tau = \widetilde{\mathsf{P}}_\tau$ and M_t is $\mathcal{F}_\tau$-measurable for any $t \in [0,\infty)$, the process M is finite and continuous also with respect to the measure $\widetilde{\mathsf{P}}$. Furthermore, as $M = (Y - \langle Y, L\rangle)^\tau$ Q-a.s., M is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. Consider the stopping times

$$\eta_n = \inf\{t \in [0,\infty)\colon |M_t| > n\}, \quad n \in \mathbb{N}.$$

Clearly, $\eta_n \uparrow \infty$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. and M^{η_n} is an $(\mathcal{F}_t, \mathsf{Q})$-martingale for any $n \in \mathbb{N}$. Since $\mathsf{Q}_\tau = \widetilde{\mathsf{P}}_\tau$, then, for any $s < t$ and $B \in \mathcal{F}_s$, we have

$$\begin{aligned} \mathsf{E}_{\widetilde{\mathsf{P}}}[I_B(M_t^{\eta_n} - M_s^{\eta_n})] &= \mathsf{E}_{\widetilde{\mathsf{P}}}[I_{B\cap\{s<\tau\}}(M_t^{\eta_n} - M_s^{\eta_n})] \\ &= \mathsf{E}_{\mathsf{Q}}[I_{B\cap\{s<\tau\}}(M_t^{\eta_n} - M_s^{\eta_n})] \\ &= \mathsf{E}_{\mathsf{Q}}[I_B(M_t^{\eta_n} - M_s^{\eta_n})] = 0. \end{aligned}$$

Hence, M is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-local martingale. Consequently, as $\widetilde{\mathsf{P}}$ is a solution of (5.5), the process

$$N_t = \int_0^{t\wedge\tau} b(X_u)\, du + \int_0^{t\wedge\tau} \beta_u\sigma^2(X_u)\, du - \int_0^{t\wedge\tau} \widetilde{b}(X_u)\, du, \quad t \in [0,\infty)$$

is well defined with respect to $\widetilde{\mathsf{P}}$ and is a continuous $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-local martingale of locally bounded variation. This means that $N = 0$ $\widetilde{\mathsf{P}}$-a.s. Thus, we have

$$\widetilde{\mathsf{P}}\left(\forall\, t \in [0,\infty)\colon \int_0^{t\wedge\tau} (b(X_u) + \beta_u\sigma^2(X_u))\, du = \int_0^{t\wedge\tau} \widetilde{b}(X_u)\, du\right) = 1.$$

As $\widetilde{\mathsf{P}}_\tau \sim \mathsf{P}_\tau$, we get

$$\mathsf{P}\left(\forall\, t \in [0,\infty), \int_0^{t\wedge\tau} (b(X_u) + \beta_u\sigma^2(X_u))\, du = \int_0^{t\wedge\tau} \widetilde{b}(X_u)\, du\right) = 1. \tag{5.42}$$

Now, let us recall that $L_t^{x_0}(X) > 0$ and $L_t^{x_0-}(X) > 0$ P-a.s. for any $t > 0$ (see [4; Th. 2.7]). Then it follows from the occupation times formula and (5.40) that, for any $t > 0$,

$$\begin{aligned} \int_0^t \frac{(\widetilde{b} - b)^2}{\sigma^2}(X_u)\, du &= \int_0^t \frac{(\widetilde{b} - b)^2}{\sigma^4}(X_u)\, d\langle X\rangle_u \\ &= \int_{\mathbb{R}} \frac{(\widetilde{b} - b)^2}{\sigma^4}(x) L_t^x(X)\, dx = \infty \quad \mathsf{P}\text{-a.s.} \end{aligned}$$

Thus,

$$\mathsf{P}\left(\forall\, t\in(0,\infty)\colon\ \int_0^t \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du=\infty\right)=1. \tag{5.43}$$

Let us recall that $\tau>0$ P-a.s. and $\int_0^t \beta_u^2\sigma^2(X_u)\,du<\infty$ P-a.s., $t\in[0,\infty)$. Therefore, conditions (5.42) and (5.43) contradict each other. As a result, $S=0$, which means that $\widetilde{\mathsf{P}}_0\perp\mathsf{P}_0$. □

Lemma 5.30. *Assume that the coefficients b, σ and $\widetilde{b}$, $\widetilde{\sigma}$ satisfy conditions* (5.2) *and* (5.3). *Let P and $\widetilde{\mathsf{P}}$ be the solutions of* (5.4) *and* (5.5) *in the sense of Definition* 5.4 *(so, we consider Setting* 2*). Let a and c be real numbers such that $-\infty<a<x_0<c<\infty$ and $[a,c]\subseteq[-\infty,\infty]\setminus A$ (recall that A denotes the complement to the set of good points). Then $\widetilde{\mathsf{P}}_{T_{a,c}}\sim\mathsf{P}_{T_{a,c}}$ and*

$$\frac{d\widetilde{\mathsf{P}}_{T_{a,c}}}{d\mathsf{P}_{T_{a,c}}}=\exp\left\{\int_0^{T_{a,c}}\frac{\widetilde{b}-b}{\sigma^2}(X_u)\,dY_u-\frac{1}{2}\int_0^{T_{a,c}}\frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du\right\}, \tag{5.44}$$

where the integrals are taken with respect to the measure P and Y is a continuous $(\mathcal{F}_t,\mathsf{P})$-local martingale defined by the formula

$$Y_t=X_{t\wedge T_{a,c}}-\int_0^{t\wedge T_{a,c}} b(X_u)\,du,\quad t\in[0,\infty).$$

`Remark.` Since P is a solution of (5.4), then Y is an $(\mathcal{F}_t,\mathsf{P})$-local martingale with the quadratic variation

$$\langle Y\rangle_t=\int_0^{t\wedge T_{a,c}}\sigma^2(X_u)\,du,\quad t\in[0,\infty).$$

Hence,

$$\int_0^{T_{a,c}}\frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du=\int_0^{T_{a,c}}\frac{(\widetilde{b}-b)^2}{\sigma^4}(X_u)\,d\langle Y\rangle_u\quad \mathsf{P}\text{-a.s.} \tag{5.45}$$

Let us show that this integral is finite P-a.s. By the occupation times formula (see [33; Ch. VI, Cor. 1.6]),

$$\begin{aligned}\int_0^{T_{a,c}}\frac{(\widetilde{b}-b)^2}{\sigma^4}(X_u)\,d\langle Y\rangle_u&=\int_0^{T_{a,c}}\frac{(\widetilde{b}-b)^2}{\sigma^4}(X_u^{T_{a,c}})\,d\langle X^{T_{a,c}}\rangle_u\\&=\int_{\mathbb{R}}\frac{(\widetilde{b}-b)^2}{\sigma^4}(x)L^x_{T_{a,c}}(X^{T_{a,c}})\,dx\quad \mathsf{P}\text{-a.s.}\end{aligned}$$

(We consider the local time of the process $X^{T_{a,c}}$ rather than of X because X may explode.) Since $[a,c]\subseteq[-\infty,\infty]\setminus A$, then $(\widetilde{b}-b)^2/\sigma^4\in L^1_{\mathrm{loc}}([a,c])$. As P-a.s. the process $(L^x_{T_{a,c}}(X^{T_{a,c}}))_{x\in\mathbb{R}}$ is equal to zero outside $[a,c]$, we have

$$\int_0^{T_{a,c}}\frac{(\widetilde{b}-b)^2}{\sigma^4}(X_u)\,d\langle Y\rangle_u<\infty\quad \mathsf{P}\text{-a.s.} \tag{5.46}$$

`Proof of Lemma 5.30.` **1)** Since A is a closed subset of $[-\infty,\infty]$, there exist a' and c' such that $-\infty<a'<a$, $c<c'<\infty$, and $[a',c']\subseteq[-\infty,\infty]\setminus A$. Let us define a continuous $(\mathcal{F}_t,\mathsf{P})$-local martingale Y' by the formula

$$Y'_t=X_{t\wedge T_{a',c'}}-\int_0^{t\wedge T_{a',c'}} b(X_u)\,du,\quad t\in[0,\infty).$$

Note that

$$\int_0^{T_{a',c'}} \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du < \infty \quad \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s.} \tag{5.47}$$

(This follows from the analogs of (5.45) and (5.46) for the process Y' instead of Y.) Fix an arbitrary $n \in \mathbb{N}$, $n > 1$. Consider the stopping time

$$\tau = \inf\left\{ t \in [0,\infty) \colon \int_0^t \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du \geq n \right\} \tag{5.48}$$

(we set $\frac{(\widetilde{b}-b)^2}{\sigma^2}(\Delta) = 0$). Consider a continuous $(\mathcal{F}_t, \mathsf{P})$-local martingale

$$L_t = \int_0^{t \wedge T_{a',c'} \wedge \tau} \frac{\widetilde{b}-b}{\sigma^2}(X_u)\,dY'_u, \quad t \in [0,\infty) \tag{5.49}$$

(L is well defined due to (5.47)). We have

$$\mathsf{E}_{\mathsf{P}} \exp\left\{ \frac{1}{2}\langle L\rangle_\infty \right\} = \mathsf{E}_{\mathsf{P}} \exp\left\{ \frac{1}{2}\int_0^{T_{a',c'}\wedge\tau} \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du \right\} \leq e^{n/2} < \infty.$$

By Novikov's criterion, the process $Z = \mathcal{E}(L)$ (i.e. Z is the stochastic exponent of L) is a uniformly integrable $(\mathcal{F}_t, \mathsf{P})$-martingale. Due to Girsanov's theorem for local martingales (see [12; Ch. III, Th. 3.11]), the process $Y' - \langle Y', L\rangle$ is a continuous $(\mathcal{F}_t, \mathsf{Q})$-local martingale, where the probability measure Q is defined by the formula $\mathsf{Q} = Z_\infty \cdot \mathsf{P}$. Note that for any $t \in [0,\infty)$,

$$Y'_t - \langle Y', L\rangle_t = X_{t\wedge T_{a',c'}} - \int_0^{t\wedge T_{a',c'}} b(X_u)\,du - \int_0^{t\wedge T_{a',c'}\wedge\tau} (\widetilde{b}-b)(X_u)\,du \quad \mathsf{Q}\text{-a.s.}$$

Consider the process

$$M_t = X_{t\wedge T_{a',c'}\wedge\tau} - \int_0^{t\wedge T_{a',c'}\wedge\tau} \widetilde{b}(X_u)\,du, \quad t \in [0,\infty). \tag{5.50}$$

It is well defined with respect to Q and $M = (Y' - \langle Y', L\rangle)^\tau$ Q-a.s. Therefore, M is a continuous $(\mathcal{F}_t, \mathsf{Q})$-local martingale with the quadratic variation

$$\langle M\rangle_t = \int_0^{t\wedge T_{a',c'}\wedge\tau} \sigma^2(X_u)\,du, \quad t \in [0,\infty).$$

Using the occupation times formula and the fact that $\sigma^2 = \widetilde{\sigma}^2$ μ_L-a.e. on $[a', c']$, we get

$$\langle M\rangle_t = \int_0^{t\wedge T_{a',c'}\wedge\tau} \widetilde{\sigma}^2(X_u)\,du, \quad t \in [0,\infty). \tag{5.51}$$

2) Let us define the functions $\widetilde{\rho}$, $\widetilde{s}$, and $\widetilde{\varkappa}$ through $\widetilde{b}$ and $\widetilde{\sigma}$ similarly to (5.6), (5.7), and (5.36). Consider the process $N = \widetilde{s}(X^{T_{a',c'}\wedge\tau})$. By the Ito-Tanaka formula (see [33; Ch. VI, Th. 1.5]) applied under the measure Q,

$$N_t = \widetilde{s}(x_0) + \int_0^t \widetilde{\rho}\big(X_u^{T_{a',c'}\wedge\tau}\big)\,dM_u, \quad t \in [0,\infty).$$

Hence, N is a continuous $(\mathcal{F}_t,\mathsf{Q})$-local martingale with the quadratic variation

$$\langle N\rangle_t=\int_0^{t\wedge T_{a',c'}\wedge\tau}\widetilde{\varkappa}^2(N_u)\,du,\quad t\in[0,\infty).$$

Since $\widetilde{\sigma}(x)\neq 0$ for any $x\in\mathbb{R}$, we have that Q-a.s. the trajectories of $\langle N\rangle$ are continuous and strictly increasing up to the time $T_{a',c'}\wedge\tau$ and they are constant after $T_{a',c'}\wedge\tau$. Let $\overline{\mathcal{F}}$ denote the Q-completion of the σ-field $\mathcal{F}$ and $(\overline{\mathcal{F}}_t)$ denote the Q-completion of the filtration $(\mathcal{F}_t)$. Define an $(\overline{\mathcal{F}}_t)$-time-change by the formula

$$\xi_t=\inf\{s\in[0,\infty)\colon \langle N\rangle_s>t\},\quad t\in[0,\infty).$$

Consider an $(\mathcal{F}'_t,\mathsf{P}')$-Brownian motion W' on a stochastic basis $(\Omega',\mathcal{F}',(\mathcal{F}'_t),\mathsf{P}')$ and set

$$\Omega=C_\Delta([0,\infty))\times\Omega',\quad \mathcal{G}=\overline{\mathcal{F}}\times\mathcal{F}',\quad \mathcal{G}_t=\bigcap_{\varepsilon>0}\overline{\mathcal{F}}_{\xi_{t+\varepsilon}}\times\mathcal{F}'_{t+\varepsilon},\quad \mathsf{R}=\mathsf{Q}\times\mathsf{P}'.$$

Denote by $\overline{\mathcal{G}}$ the R-completion of the σ-field $\mathcal{G}$ and by $(\overline{\mathcal{G}}_t)$ the R-completion of the filtration $(\mathcal{G}_t)$. Consider the stochastic basis $(\Omega,\overline{\mathcal{G}},(\overline{\mathcal{G}}_t),\mathsf{R})$. All the random variables and the processes defined on $C_\Delta([0,\infty))$ or on Ω' can be viewed as random variables and processes on Ω. In what follows, we do not explain on which space we consider a random variable or a process if this is clear from the context.

Set

$$W_t=N_{\xi_t}+W'_t-W'_{t\wedge\langle N\rangle_\infty},\quad t\in[0,\infty).$$

By the Dambis-Dubins-Schwartz theorem (see [33; Ch. V, Th. 1.6]), the process $W=(W_t)_{t\in[0,\infty)}$ is a $(\overline{\mathcal{G}}_t,\mathsf{R})$-Brownian motion with the starting point $\widetilde{s}(x_0)$.

As Q-a.s. the trajectories of $\langle N\rangle$ are continuous, we have

$$\langle N\rangle_{\xi_t}=t\quad \mathsf{Q}\text{-a.s. on the set } \{t<\langle N\rangle_\infty\},$$

i.e.

$$\int_0^{\xi_t}\widetilde{\varkappa}^2(N_u)\,du=t\quad \mathsf{Q}\text{-a.s. on the set } \{t<\langle N\rangle_\infty\}.$$

As Q-a.s. the trajectories of $\langle N\rangle$ are strictly increasing up to the time $T_{a',c'}\wedge\tau$, we have that Q-a.s. the trajectories of ξ are continuous up to the time $\langle N\rangle_\infty$. By the change of variables in the Stieltjes integral, we get

$$\int_0^t\widetilde{\varkappa}^2(N_{\xi_u})\,d\xi_u=t\quad \mathsf{Q}\text{-a.s. on the set } \{t<\langle N\rangle_\infty\},$$

and hence,

$$\xi_t=\int_0^t\widetilde{\varkappa}^{-2}(N_{\xi_u})\,du\quad \mathsf{Q}\text{-a.s. on the set } \{t<\langle N\rangle_\infty\}.$$

Clearly, $\xi_t=\infty$ whenever $t\ge\langle N\rangle_\infty$. Therefore, R-a.s. for any $t\in[0,\infty)$,

$$\xi_t=\begin{cases}\int_0^t\widetilde{\varkappa}^{-2}(W_u)\,du & \text{if } t<\langle N\rangle_\infty,\\ \infty & \text{if } t\ge\langle N\rangle_\infty.\end{cases}$$

Using the occupation times formula, it is easy to verify that P-a.s. we have

$$\forall\, t < \langle N\rangle_\infty,\quad \int_0^{\xi_t} \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du = \int_0^{\xi_t} \frac{(\widetilde{b}-b)^2}{\widetilde{\sigma}^2}(X_u)\,du.$$

By the change of variables in the Stieltjes integral, R-a.s. we get

$$\begin{aligned}\forall\, t < \langle N\rangle_\infty,\quad \int_0^{\xi_t} \frac{(\widetilde{b}-b)^2}{\widetilde{\sigma}^2}(X_u)\,du &= \int_0^{\xi_t} \frac{(\widetilde{b}-b)^2}{\widetilde{\sigma}^2}(\widetilde{s}^{\,-1}(N_u))\,du \\ &= \int_0^{t} \frac{(\widetilde{b}-b)^2}{\widetilde{\sigma}^2}(\widetilde{s}^{\,-1}(N_{\xi_u}))\,d\xi_u \qquad (5.52)\\ &= \int_0^{t} \frac{(\widetilde{b}-b)^2}{\widetilde{\rho}^2\widetilde{\sigma}^4}(\widetilde{s}^{\,-1}(W_u))\,du.\end{aligned}$$

Letting $t \uparrow \langle N\rangle_\infty$ in (5.52), we get

$$\int_0^{T_{a',c'}\wedge\tau} \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du = \int_0^{\langle N\rangle_\infty} \frac{(\widetilde{b}-b)^2}{\widetilde{\rho}^2\widetilde{\sigma}^4}(\widetilde{s}^{\,-1}(W_u))\,du \quad \text{R-a.s.} \qquad (5.53)$$

Set

$$\eta(W) = \inf\left\{ t \in [0,\infty)\colon\ \int_0^{t} \frac{(\widetilde{b}-b)^2}{\widetilde{\rho}^2\widetilde{\sigma}^4}(\widetilde{s}^{\,-1}(W_u))\,du \ge n \right\}$$

(we set $\frac{(\widetilde{b}-b)^2}{\widetilde{\rho}^2\widetilde{\sigma}^4}(\widetilde{s}^{\,-1}(x)) = 0$ if $x \notin \widetilde{s}(\mathbb{R})$), where n is the number that appears in (5.48). Let us now prove the equality

$$\langle N\rangle_\infty = T_{\widetilde{s}(a'),\widetilde{s}(c')}(W) \wedge \eta(W) \quad \text{R-a.s.} \qquad (5.54)$$

For this, note that

$$\int_0^{T_{a',c'}\wedge\tau} \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du = n \quad \text{P-a.s. on the set } \{\tau < T_{a',c'}\}. \qquad (5.55)$$

Indeed, condition (5.55) may be violated only if the integral is less than n and the process $\big(\int_0^t \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du\big)_{t\in[0,\infty)}$ jumps to infinity at time τ. But P-a.s. this cannot happen on the set $\{\tau < T_{a',c'}\}$ since (5.47) holds. Moreover, as $\xi_{\langle N\rangle_\infty -} = T_{a',c'} \wedge \tau$, we have $\langle N\rangle_\infty \ge T_{\widetilde{s}(a'),\widetilde{s}(c')}(W)$ R-a.s. on the set $\{T_{a',c'} \le \tau\}$. By (5.53) and (5.55), $\langle N\rangle_\infty \ge \eta(W)$ R-a.s. on the set $\{\tau < T_{a',c'}\}$. Thus, $\langle N\rangle_\infty \ge T_{\widetilde{s}(a'),\widetilde{s}(c')}(W) \wedge \eta(W)$ R-a.s. Finally, the reverse inequality easily follows from (5.52). So, statement (5.54) is proved.

It follows from the reasoning above that R-a.s. for any $t \in [0,\infty)$,

$$\xi_t = \begin{cases} \displaystyle\int_0^t \widetilde{\varkappa}^{-2}(W_u)\,du & \text{if } t < T_{\widetilde{s}(a'),\widetilde{s}(c')}(W) \wedge \eta(W),\\ \infty & \text{if } t \ge T_{\widetilde{s}(a'),\widetilde{s}(c')}(W) \wedge \eta(W).\end{cases}$$

Let us recall that

$$\begin{aligned}&\langle N\rangle_t = \inf\{s \in [0,\infty)\colon \xi_s > t\} \quad \text{Q-a.s.},\quad t \in [0,\infty),\\ &N_t = W_{\langle N\rangle_t} \quad \text{R-a.s.},\quad t \in [0,\infty),\\ &X^{T_{a',c'}\wedge\tau} = \widetilde{s}^{\,-1}(N).\end{aligned}$$

So, we obtain an explicit construction of the measure $\operatorname{Law}\big(X^{T_{a',c'}\wedge\tau}\big|\mathsf{Q}\big)$ through the Wiener measure. Furthermore, as $\widetilde{\mathsf{P}}$ is a solution of (5.5), the process M introduced in (5.50) is a continuous $(\mathcal{F}_t,\widetilde{\mathsf{P}})$-local martingale with the same quadratic variation as in formula (5.51). Therefore, repeating the reasoning of part 2) with the measure $\widetilde{\mathsf{P}}$ instead of Q, we obtain that the measure $\operatorname{Law}\big(X^{T_{a',c'}\wedge\tau}\big|\widetilde{\mathsf{P}}\big)$ can be constructed from the Wiener measure in the same way as $\operatorname{Law}\big(X^{T_{a',c'}\wedge\tau}\big|\mathsf{Q}\big)$. Thus,

$$\operatorname{Law}\big(X^{T_{a',c'}\wedge\tau}\big|\widetilde{\mathsf{P}}\big) = \operatorname{Law}\big(X^{T_{a',c'}\wedge\tau}\big|\mathsf{Q}\big). \tag{5.56}$$

3) Consider the stopping time

$$\rho = \inf\left\{t\in[0,\infty)\colon \int_0^t \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du \ge n-1\right\},$$

where n appears in (5.48). Using (5.55) and the analogous condition for the measure $\widetilde{\mathsf{P}}$, we get $T_{a,c}\wedge\rho < T_{a',c'}\wedge\tau$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s. Applying Lemma 5.22, we obtain that $\mathsf{P},\widetilde{\mathsf{P}}$-a.s. for any event $B\in\mathcal{F}_{T_{a,c}\wedge\rho}$,

$$X\in B \iff X^{T_{a',c'}\wedge\tau}\in B.$$

Then, due to (5.56), for any $B\in\mathcal{F}_{T_{a,c}\wedge\rho}$, we have

$$\widetilde{\mathsf{P}}(B) = \widetilde{\mathsf{P}}(X\in B) = \widetilde{\mathsf{P}}(X^{T_{a',c'}\wedge\tau}\in B) = \mathsf{Q}(X^{T_{a',c'}\wedge\tau}\in B) = \mathsf{Q}(X\in B) = \mathsf{Q}(B).$$

Consequently, the measures Q and $\widetilde{\mathsf{P}}$ coincide on the σ-field $\mathcal{F}_{T_{a,c}\wedge\rho}$. Let us now recall that $\mathsf{Q} = Z_\infty\cdot\mathsf{P}$, where the uniformly integrable $(\mathcal{F}_t,\mathsf{P})$-martingale Z is defined by the formula $Z=\mathcal{E}(L)$ and L is defined in (5.49). Hence, $\widetilde{\mathsf{P}}_{T_{a,c}\wedge\rho}\sim\mathsf{P}_{T_{a,c}\wedge\rho}$ and

$$\frac{d\widetilde{\mathsf{P}}_{T_{a,c}\wedge\rho}}{d\mathsf{P}_{T_{a,c}\wedge\rho}} = \mathsf{E}_{\mathsf{P}}(Z_\infty|\mathcal{F}_{T_{a,c}\wedge\rho}) = Z_{T_{a,c}\wedge\rho}. \tag{5.57}$$

4) Now, let us use the notation

$$\tau_n = \inf\left\{t\in[0,\infty)\colon \int_0^t \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du \ge n\right\},\qquad n\in\mathbb{N}.$$

(We fixed some $n\in\mathbb{N}$ above and considered stopping times τ_n and τ_{n-1}, which were denoted by τ and ρ for the simplicity of notation. Below we need to use all τ_n. That is why we now change the notation.) By (5.57),

$$\frac{d\widetilde{\mathsf{P}}_{T_{a,c}\wedge\tau_n}}{d\mathsf{P}_{T_{a,c}\wedge\tau_n}} = \exp\left\{\int_0^{T_{a,c}\wedge\tau_n}\frac{\widetilde{b}-b}{\sigma^2}(X_u)dY'_u - \frac{1}{2}\int_0^{T_{a,c}\wedge\tau_n}\frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)du\right\}. \tag{5.58}$$

It follows from (5.47) that

$$\lim_{n\to\infty}\tau_n \ge T_{a',c'} > T_{a,c}\quad \mathsf{P},\widetilde{\mathsf{P}}\text{-a.s.} \tag{5.59}$$

As a consequence, we get

$$\mathcal{F}_{T_{a,c}} = \bigvee_{n=1}^{\infty}\mathcal{F}_{T_{a,c}\wedge\tau_n} \tag{5.60}$$

up to events of $\mathsf{P}, \widetilde{\mathsf{P}}$-zero measure. (Indeed, the inclusion $\mathcal{F}_{T_{a,c}} \subseteq \bigvee_{n=1}^{\infty} \mathcal{F}_{T_{a,c}\wedge\tau_n}$ follows from the formula

$$B = \bigcup_{n=1}^{\infty} (B \cap \{T_{a,c} = T_{a,c} \wedge \tau_n\}) \quad \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s.},$$

and the reverse inclusion is obvious.) Formulas (5.58), (5.59), and (5.60) imply that

$$\frac{d\widetilde{\mathsf{P}}_{T_{a,c}}}{d\mathsf{P}_{T_{a,c}}} = \exp\left\{ \int_0^{T_{a,c}} \frac{\widetilde{b} - b}{\sigma^2}(X_u)\, dY'_u - \frac{1}{2} \int_0^{T_{a,c}} \frac{(\widetilde{b} - b)^2}{\sigma^2}(X_u)\, du \right\}, \tag{5.61}$$

where by $\frac{d\widetilde{\mathsf{P}}_{T_{a,c}}}{d\mathsf{P}_{T_{a,c}}}$ we denote the density of the absolutely continuous part of the measure $\widetilde{\mathsf{P}}_{T_{a,c}}$ with respect to the measure $\mathsf{P}_{T_{a,c}}$. Since $\frac{d\widetilde{\mathsf{P}}_{T_{a,c}}}{d\mathsf{P}_{T_{a,c}}} > 0$ P-a.s., we get $\mathsf{P}_{T_{a,c}} \ll \widetilde{\mathsf{P}}_{T_{a,c}}$. Due to the symmetry between P and $\widetilde{\mathsf{P}}$, $\widetilde{\mathsf{P}}_{T_{a,c}} \ll \mathsf{P}_{T_{a,c}}$. Thus, $\widetilde{\mathsf{P}}_{T_{a,c}} \sim \mathsf{P}_{T_{a,c}}$ and the density of $\widetilde{\mathsf{P}}_{T_{a,c}}$ with respect to $\mathsf{P}_{T_{a,c}}$ is given by formula (5.61). Finally, it is clear that the process Y' in (5.61) may be replaced by Y. □

Before passing on to the proof of Theorem 5.7, we need one more technical lemma.

Lemma 5.31. *In Setting 2, consider $a \in \mathbb{R}$ and a sequence (c_n) such that $c_1 > a$, $c_{n+1} > c_n$, and $c_n \uparrow \infty$. Then $\mathcal{F}_{T_a} = \bigvee_{n=1}^{\infty} \mathcal{F}_{T_{a,c_n}}$.*

`Proof.` Consider the collection $\mathcal{D}$ of sets $B \in \mathcal{F}$ such that

$$B \cap \{T_a = \infty, \overline{\lim_{t\uparrow\zeta}} X_t = \infty\} \in \bigvee_{n=1}^{\infty} \mathcal{F}_{T_{a,c_n}}.$$

Notice that

$$\{T_a = \infty, \overline{\lim_{t\uparrow\zeta}} X_t = \infty\} = \bigcap_{n=1}^{\infty} \{X_{T_{a,c_n}} I(T_{a,c_n} < \infty) = c_n\} \in \bigvee_{n=1}^{\infty} \mathcal{F}_{T_{a,c_n}}. \tag{5.62}$$

Now, one can easily check that $\mathcal{D}$ is a σ-field. Since for any $t \in [0, \infty)$ and $d \in \mathbb{R}$,

$$\begin{aligned} &\{X_t < d\} \cap \{T_a = \infty, \overline{\lim_{t\uparrow\zeta}} X_t = \infty\} \\ &= \left[\bigcup_{n=1}^{\infty} \big(\{T_{a,c_n} > t\} \cap \{X_{t\wedge T_{a,c_n}} < d\}\big) \right] \cap \{T_a = \infty, \overline{\lim_{t\uparrow\zeta}} X_t = \infty\}, \end{aligned}$$

then by applying (5.62), we obtain $\mathcal{D} = \sigma(X_t;\, t \in [0, \infty)) = \mathcal{F}$.

Now, the inclusion $\mathcal{F}_{T_a} \subseteq \bigvee_{n=1}^{\infty} \mathcal{F}_{T_{a,c_n}}$ follows from the formula

$$B = \left[\bigcup_{n=1}^{\infty} \big(B \cap \{T_a = T_{a,c_n}\}\big) \right] \cup \big(B \cap \{T_a = \infty, \overline{\lim_{t\uparrow\zeta}} X_t = \infty\}\big),$$

and the reverse inclusion is obvious. □

`Proof of Theorem 5.7.` We should prove only (ii). Therefore, below we assume that $\mathsf{P} \neq \widetilde{\mathsf{P}}$. Set

$$\tau = \sup_n \overline{\inf}\{t \in [0, \infty)\colon X_t \in A^{1/n}\}.$$

Let us prove that the separating time S equals τ. Denote by α the "bad point that is closest to x_0 from the left side" (see (5.14)). Similarly, denote by γ the "bad pint that is closest to x_0 from the right side". It is convenient for us to set

$$\alpha' = \begin{cases} -\infty & \text{if } \alpha = \Delta, \\ \alpha & \text{if } \alpha \neq \Delta \end{cases}$$

and

$$\gamma' = \begin{cases} \infty & \text{if } \gamma = \Delta, \\ \gamma & \text{if } \gamma \neq \Delta. \end{cases}$$

If $x_0 \notin A$ (or, equivalently, $\alpha' < x_0 < \gamma'$), then we consider sequences (a_n) and (c_n) such that $a_1 < x_0 < c_1$, $a_{n+1} < a_n$, $a_n \downarrow \alpha'$, $c_{n+1} > c_n$, and $c_n \uparrow \gamma'$.

The proof consists of two parts.

I. Let us first prove that $S \geq \tau$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. If $x_0 \in A$, then $\tau = 0$ and this inequality is obvious. Therefore, we consider the case $x_0 \notin A$. By Lemma 5.30, $\widetilde{\mathsf{P}}_{T_{a_n,c_n}} \sim \mathsf{P}_{T_{a_n,c_n}}$ for any $n \in \mathbb{N}$, and hence, $S > T_{a_n,c_n}$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.

Suppose that $\alpha \neq \Delta$ and $\gamma \neq \Delta$. Clearly, in this case $T_{a_n,c_n} \uparrow \tau$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.. Thus, we obtain the desired inequality $S \geq \tau$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.

Suppose now that $\alpha = \Delta$ or $\gamma = \Delta$. In this case $T_{a_n,c_n} \uparrow \tau \wedge \zeta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s., and hence,

$$S \geq \tau \wedge \zeta \quad \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s.} \tag{5.63}$$

It is easy to establish that

$$\{\tau > \zeta\} = B_{-\infty} \cup B_{\infty} \quad \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s.}, \tag{5.64}$$

where

$$B_{-\infty} = \begin{cases} \{\lim_{t \uparrow \zeta} X_t = -\infty\} \cap \{\forall\, t < \zeta \colon X_t < \gamma'\} & \text{if } \alpha = \Delta, \\ \emptyset & \text{if } \alpha \neq \Delta, \end{cases}$$

$$B_{\infty} = \begin{cases} \{\lim_{t \uparrow \zeta} X_t = \infty\} \cap \{\forall\, t < \zeta \colon X_t > \alpha'\} & \text{if } \gamma = \Delta, \\ \emptyset & \text{if } \gamma \neq \Delta. \end{cases}$$

Let us prove that $\widetilde{\mathsf{P}} \sim \mathsf{P}$ on the set B_∞. If $\gamma \neq \Delta$, then this is obvious. Therefore, we consider the case $\gamma = \Delta$. Fix $a \in (\alpha', x_0)$ and define continuous $(\mathcal{F}_t, \mathsf{P})$-local martingales Y^n, L^n, and Z^n by the formulas

$$Y_t^n = X_{t \wedge T_{a,c_n}} - \int_0^{t \wedge T_{a,c_n}} b(X_u)\, du, \qquad t \in [0, \infty),$$

$$L_t^n = \int_0^{t \wedge T_{a,c_n}} \frac{\widetilde{b} - b}{\sigma^2}(X_u)\, dY_u^n, \qquad t \in [0, \infty),$$

$$Z_t^n = \exp\left\{ L_t^n - \frac{1}{2} \langle L^n \rangle_t \right\}, \qquad t \in [0, \infty).$$

Note that the process L^n is well defined with respect to the measure P (see the Remark following Lemma 5.30). Clearly, $Z^n = \mathcal{E}(L^n)$ (i.e. Z^n is the stochastic exponent of L^n).

Set $T = T_a \wedge \zeta$. Since $T_{a,c_n} \uparrow T$ P-a.s. and

$$L_t^{n+1} = L_t^n \quad \mathsf{P}\text{-a.s. on the set } \{t < T_{a,c_n}\},$$
$$Z_t^{n+1} = Z_t^n \quad \mathsf{P}\text{-a.s. on the set } \{t < T_{a,c_n}\},$$

we can define continuous $(\mathcal{F}_t, \mathsf{P})$-local martingales L and Z on the stochastic interval $[0,T)$ (for the definition of a process on a stochastic interval, see [33; Ch. IV, Ex. 1.48]) such that

$$L_t = L_t^n \quad \mathsf{P}\text{-a.s. on the set } \{t < T_{a,c_n}\},$$
$$Z_t = Z_t^n \quad \mathsf{P}\text{-a.s. on the set } \{t < T_{a,c_n}\}.$$

Notice that

$$Z_t = \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\}, \quad t \in [0,T)$$

and

$$\langle L\rangle_t = \int_0^t \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du, \quad t \in [0,T).$$

Since Z is positive, it converges P-a.s. as $t \uparrow T$ to a finite random variable Z_T (this follows from the Dambis-Dubins-Schwartz theorem for continuous local martingales on a stochastic interval; see [33; Ch. V, Ex. 1.18]). Hence, $Z_{T_{a,c_n}} \to Z_T$ P-a.s. Furthermore, due to Lemma 5.30, $Z_{T_{a,c_n}} = \frac{d\widetilde{\mathsf{P}}_{T_{a,c_n}}}{d\mathsf{P}_{T_{a,c_n}}}$, and due to Lemma 5.31, $\mathcal{F}_{T_a} = \bigvee_{n=1}^{\infty} \mathcal{F}_{T_{a,c_n}}$. By the Jessen theorem (see [42; Th. 5.2.26]), Z_T is the density of the absolutely continuous part of the measure $\widetilde{\mathsf{P}}_{T_a}$ with respect to the measure P_{T_a}.

Applying Lemma 5.27 to the function $f = (\widetilde{b}-b)^2/\sigma^2$, we get $\langle L\rangle_T < \infty$ P-a.s. on the set $\{T_a = \infty\}$ (recall that we consider the case $\gamma = \Delta$, i.e. ∞ is a good point). Clearly, $\langle L\rangle_T < \infty$ P-a.s. on the set $\{T_a < \infty\}$. Hence, $\langle L\rangle_T < \infty$ P-a.s. It follows now from the Dambis-Dubins-Schwartz theorem for continuous local martingales on a stochastic interval that $Z_T > 0$ P-a.s. Consequently, $\mathsf{P}_{T_a} \ll \widetilde{\mathsf{P}}_{T_a}$.

Since ∞ is a good point, $s(\infty) < \infty$. By Proposition A.3, $\mathsf{P}(T_a = \infty) > 0$. As $\mathsf{P}_{T_a} \ll \widetilde{\mathsf{P}}_{T_a}$, we get $\widetilde{\mathsf{P}}(T_a = \infty) > 0$. Hence, $\widetilde{s}(\infty) < \infty$. Now, let us prove that the condition

$$(\widetilde{s}(\infty) - \widetilde{s})\frac{(b-\widetilde{b})^2}{\widetilde{\rho}\,\widetilde{\sigma}^4} \in L^1_{\mathrm{loc}}(\infty). \tag{5.65}$$

holds. For this, apply the above reasoning to P instead of $\widetilde{\mathsf{P}}$. Define continuous $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-local martingales $\widetilde{L}$ and $\widetilde{Z}$ on the stochastic interval $[0,T)$ similarly to the processes L and Z. Then $\widetilde{Z}_T$ is the density of the absolutely continuous part of the measure P_{T_a} with respect to the measure $\widetilde{\mathsf{P}}_{T_a}$. If condition (5.65) does not hold, then, by Lemma 5.27, $\langle \widetilde{L}\rangle_T = \infty$ $\widetilde{\mathsf{P}}$-a.s. on the set $\{T_a = \infty\}$. Due to the Dambis-Dubins-Schwartz theorem for continuous local martingales on a stochastic interval, we have $\underline{\lim}_{t\uparrow T} \widetilde{L}_t = -\infty$ $\widetilde{\mathsf{P}}$-a.s. on the set $\{T_a = \infty\}$. Hence, $\widetilde{\mathsf{P}}$-a.s. on the set $\{T_a = \infty\}$ we get

$$\widetilde{Z}_T = \underline{\lim}_{t\uparrow T} \widetilde{Z}_t = \exp\left\{\underline{\lim}_{t\uparrow T} \widetilde{L}_t - \frac{1}{2}\langle \widetilde{L}\rangle_T\right\} = 0.$$

As a consequence, $\widetilde{\mathsf{P}}_{T_a} \perp \mathsf{P}_{T_a}$ on the set $\{T_a = \infty\}$, which contradicts the conditions $\mathsf{P}_{T_a} \ll \widetilde{\mathsf{P}}_{T_a}$ and $\mathsf{P}(T_a = \infty) > 0$. Hence, condition (5.65) holds.

Since $\widetilde{s}(\infty) < \infty$ and condition (5.65) holds, we can repeat the above reasoning using the processes $\widetilde{L}$ and $\widetilde{Z}$ instead of L and Z. As a result, we get $\widetilde{Z}_T > 0$ $\widetilde{\mathsf{P}}$-a.s., and therefore, $\widetilde{\mathsf{P}}_{T_a} \ll \mathsf{P}_{T_a}$.

Thus, $\widetilde{\mathsf{P}}_{T_a} \sim \mathsf{P}_{T_a}$. Hence, $\widetilde{\mathsf{P}} \sim \mathsf{P}$ on the set $\{T_a = \infty\}$. Since $a \in (\alpha', x_0)$ is arbitrary, and in view of the fact that the sets $\{T_a = \infty\}$ tend to B_∞ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. as $a \downarrow \alpha'$, we get that $\widetilde{\mathsf{P}} \sim \mathsf{P}$ on the set B_∞. Similarly, $\widetilde{\mathsf{P}} \sim \mathsf{P}$ on the set $B_{-\infty}$. Consequently, $S = \delta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. on the set $B_{-\infty} \cup B_\infty$. Combining this with (5.63) and (5.64), we obtain the desired inequality $S \geq \tau$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.

II. Let us now prove that $S \leq \tau$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. Consider several cases.

1) Suppose that $x_0 \in A$. Set

$$\begin{aligned}
b'(x) &= b(x) I_{[x_0-2, x_0+2]}(x), \quad x \in \mathbb{R},\\
b''(x) &= \widetilde{b}(x) I_{[x_0-2, x_0+2]}(x), \quad x \in \mathbb{R},\\
\sigma'(x) &= \sigma(x), \quad x \in \mathbb{R},\\
\sigma''(x) &= \widetilde{\sigma}(x), \quad x \in \mathbb{R}
\end{aligned}$$

and consider the SDEs

$$dX_t = b'(X_t)\,dt + \sigma'(X_t)\,dB_t, \quad X_0 = x_0, \tag{5.66}$$

$$dX_t = b''(X_t)\,dt + \sigma''(X_t)\,dB_t, \quad X_0 = x_0. \tag{5.67}$$

The coefficients b', σ' and b'', σ'' satisfy conditions (5.2) and (5.3). Let P' and P'' denote the solutions of (5.66) and (5.67) in the sense of Definition 5.4. By [4; Th. 2.11], $\mathsf{P}_{T_{x_0-1,x_0+1}} = \mathsf{P}'_{T_{x_0-1,x_0+1}}$ and $\widetilde{\mathsf{P}}_{T_{x_0-1,x_0+1}} = \mathsf{P}''_{T_{x_0-1,x_0+1}}$. It follows from Propositions A.1 and A.2 that P' and P'' do not explode. Due to Lemma 5.29, $\mathsf{P}'_0 \perp \mathsf{P}''_0$. Therefore, $\widetilde{\mathsf{P}}_0 \perp \mathsf{P}_0$, and hence $S = 0$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.

2) Suppose that $-\infty < \alpha < x_0 < \gamma < \infty$. Then $\tau = T_{\alpha,\gamma}$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. Since $T_{\alpha,\gamma} < \infty$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s., then, using the strong Markov property of solutions of SDEs (see [43; Th. 6.2] or [18; Th. 18.11]) and the result of 1), we obtain that $\widetilde{\mathsf{P}}_{T_{\alpha,\gamma}} \perp \mathsf{P}_{T_{\alpha,\gamma}}$. Hence, $S \leq T_{\alpha,\gamma} = \tau$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.

3) Suppose that $-\infty < \alpha < x_0$, $\gamma = \infty$. Then $\tau = T_\alpha \wedge \zeta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. Therefore, we need to prove that

$$S \leq T_\alpha \quad \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s. on the set } \{T_\alpha < \infty\} \tag{5.68}$$

and

$$S \leq \zeta \quad \mathsf{P}, \widetilde{\mathsf{P}}\text{-a.s. on the set } \{T_\alpha = \infty\}. \tag{5.69}$$

Condition (5.68) holds due to the strong Markov property of solutions of SDEs. Prior to proving (5.69), let us notice that $\mathcal{F}_\zeta = \mathcal{F}$. Hence, $\mathcal{F}_{T_\alpha \wedge \zeta} = \mathcal{F}_{T_\alpha} \cap \mathcal{F}_\zeta = \mathcal{F}_{T_\alpha}$.

If $s(\infty) = \infty$, then $\mathsf{P}(T_\alpha = \infty) = 0$. Therefore, $\widetilde{\mathsf{P}}_{T_\alpha \wedge \zeta} \perp \mathsf{P}_{T_\alpha \wedge \zeta}$ on the set $\{T_\alpha = \infty\}$. Consequently, $S \leq T_\alpha \wedge \zeta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. on the set $\{T_\alpha = \infty\}$ and it follows that (5.69) holds.

Finally, let us prove (5.69) in the case, where $s(\infty) < \infty$. For this, fix $a \in (\alpha, x_0)$, set $T = T_a \wedge \zeta$, and consider the continuous $(\mathcal{F}_t, \mathsf{P})$-local martingales L and Z on the stochastic interval $[0, T)$ introduced in part I of the proof. By Lemma 5.27, $\langle L \rangle_T = \infty$ P-a.s. on the set $\{T_a = \infty\}$ (recall that here ∞ is a bad point). Due to the Dambis-Dubins-Schwartz theorem for continuous local martingales on a stochastic interval, we have $\underline{\lim}_{t\uparrow T} L_t = -\infty$ P-a.s. on the set $\{T_a = \infty\}$. Hence, P-a.s. on the set $\{T_a = \infty\}$ we get

$$Z_T = \underline{\lim}_{t\uparrow T} Z_t = \exp\left\{\underline{\lim}_{t\uparrow T} L_t - \frac{1}{2}\langle L\rangle_T\right\} = 0.$$

Since Z_T is the density of the absolutely continuous part of the measure $\widetilde{\mathsf{P}}_{T_a}$ with respect to the measure P_{T_a}, we have $\widetilde{\mathsf{P}}_{T_a} \perp \mathsf{P}_{T_a}$ on the set $\{T_a = \infty\}$. As $\mathcal{F}_{T_a\wedge\zeta} = \mathcal{F}_{T_a}$, we get $\widetilde{\mathsf{P}}_{T_a\wedge\zeta} \perp \mathsf{P}_{T_a\wedge\zeta}$ on the set $\{T_a = \infty\}$. Hence, $S \le T_a \wedge \zeta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. on the set $\{T_a = \infty\}$. Since $a \in (\alpha, x_0)$ is arbitrary, condition (5.69) is satisfied.

In a similar way, we consider the case, where $\alpha = -\infty$, $x_0 < \gamma < \infty$.

4) Suppose that $-\infty < \alpha < x_0$, $\gamma = \Delta$. Then $\tau = \inf\{t \in [0,\infty) \colon X_t = \alpha\}$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. Therefore, we need to prove only condition (5.68), and this follows from the strong Markov property of solutions of SDEs.

In a similar way, we consider the case, where $\alpha = \Delta$, $x_0 < \gamma < \infty$.

5) Suppose that $\alpha = -\infty$, $\gamma = \infty$. Then $\tau = \zeta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. Let us first assume that $s(-\infty) > -\infty$ or $s(\infty) < \infty$. It follows from Propositions A.2 and A.3 that in this case

$$\mathsf{P}(\{\lim_{t\uparrow\zeta} X_t = \infty\} \cup \{\lim_{t\uparrow\zeta} X_t = -\infty\}) = 1. \tag{5.70}$$

Similarly to the proof of (5.69), we establish that $S \le \zeta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. on the set $\{\lim_{t\uparrow\zeta} X_t = \infty\}$ and $S \le \zeta$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s. on the set $\{\lim_{t\uparrow\zeta} X_t = -\infty\}$. Hence, by (5.70), $\widetilde{\mathsf{P}} \perp \mathsf{P}$. Since $\mathcal{F}_\zeta = \mathcal{F}$, we have $\widetilde{\mathsf{P}}_\zeta \perp \mathsf{P}_\zeta$. Thus, $S \le \zeta = \tau$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.

Let us now assume that $s(-\infty) = -\infty$ and $s(\infty) = \infty$. Then the measure P does not explode. Consider the continuous $(\mathcal{F}_t, \mathsf{P})$-local martingale

$$Y_t = X_t - \int_0^t b(X_u)\,du, \quad t \in [0,\infty).$$

By the occupation times formula (see [33; Ch. VI, Cor. 1.6]),

$$\begin{aligned}\int_0^t \frac{(\widetilde{b}-b)^2}{\sigma^4}(X_u)\,d\langle Y\rangle_u &= \int_0^t \frac{(\widetilde{b}-b)^2}{\sigma^4}(X_u)\,d\langle X\rangle_u \\ &= \int_{\mathbb{R}} \frac{(\widetilde{b}-b)^2}{\sigma^4}(x) L_t^x(X)\,dx < \infty \quad \mathsf{P}\text{-a.s.},\end{aligned}$$

since P-a.s. the process $(L_t^x(X))_{x\in\mathbb{R}}$ is equal to zero outside a finite interval (let us recall that in the case under consideration, $(\widetilde{b}-b)^2/\sigma^4 \in L^1_{\mathrm{loc}}(\mathbb{R})$). Hence, the continuous $(\mathcal{F}_t, \mathsf{P})$-local martingales

$$L_t = \int_0^t \frac{\widetilde{b}-b}{\sigma^2}(X_u)\,dY_u, \quad t \in [0,\infty)$$

and

$$Z_t = \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\}, \quad t \in [0,\infty)$$

are well defined with respect to the measure P (note that $Z = \mathcal{E}(L)$). Since Z is a positive $(\mathcal{F}_t, \mathsf{P})$-local martingale, it converges P-a.s. as $t \to \infty$ to a finite random variable Z_∞. Consider sequences (a_n) and (c_n) such that $a_1 < x_0 < c_1$, $a_{n+1} < a_n$, $a_n \downarrow -\infty$,

$c_{n+1} > c_n$, and $c_n \uparrow \infty$. Then $Z_{T_{a_n,c_n}} \to Z_\infty$ P-a.s. By Lemma 5.30, $Z_{T_{a_n,c_n}} = \frac{d\widetilde{\mathsf{P}}_{T_{a_n,c_n}}}{d\mathsf{P}_{T_{a_n,c_n}}}$. By the Jessen theorem (see [42; Th. 5.2.26]), Z_∞ is the density of the absolutely continuous part of the measure $\widetilde{\mathsf{Q}}$ with respect to the measure Q, where Q and $\widetilde{\mathsf{Q}}$ are the restrictions of P and $\widetilde{\mathsf{P}}$ to the σ-field $\bigvee_{n=1}^\infty \mathcal{F}_{T_{a_n,c_n}}$.

Due to Lemma 5.28,

$$\langle L\rangle_\infty = \int_0^\infty \frac{(\widetilde{b}-b)^2}{\sigma^2}(X_u)\,du = \infty \quad \mathsf{P}\text{-a.s.}$$

Consequently,

$$Z_\infty = \lim_{t\to\infty} Z_t = \exp\left\{\lim_{t\to\infty} L_t - \frac{1}{2}\langle L\rangle_\infty\right\} = 0 \quad \mathsf{P}\text{-a.s.}$$

Hence, $\widetilde{\mathsf{Q}} \perp \mathsf{Q}$, i.e. $\widetilde{\mathsf{P}} \perp \mathsf{P}$. Since $\mathcal{F}_\zeta = \mathcal{F}$, we have $\widetilde{\mathsf{P}}_\zeta \perp \mathsf{P}_\zeta$. Thus, $S \le \zeta = \tau$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s.

6) Suppose that $\alpha = \Delta$, $\gamma = \infty$. Consider the sets

$$D = \bigl\{\zeta = \infty,\ \overline{\lim_{t\to\infty}} X_t = \infty,\ \underline{\lim_{t\to\infty}} X_t = -\infty\bigr\},$$
$$D_+ = \bigl\{\lim_{t\uparrow\zeta} X_t = \infty\bigr\}, \quad D_- = \bigl\{\lim_{t\uparrow\zeta} X_t = -\infty\bigr\}.$$

By Proposition A.1,

$$\mathsf{P}(D \cup D_+ \cup D_-) = \widetilde{\mathsf{P}}(D \cup D_+ \cup D_-) = 1.$$

In the case under consideration, $\tau = \delta$ on D_- $\tau = \infty$ on the set D, $\tau = \zeta$ on the set D_+. Since $s(-\infty) > -\infty$ ($-\infty$ is a good point), we have $\mathsf{P}(D) = 0$. Consequently, $\widetilde{\mathsf{P}} \perp \mathsf{P}$ on the set D, and therefore, $S \le \infty$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s. on the set D. Similarly to the proof of (5.69), we establish that $S \le \zeta$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s on the set D_+. Thus, $S \le \tau$ $\mathsf{P},\widetilde{\mathsf{P}}$-a.s.

In a similar way, we consider the case, where $\alpha = -\infty$, $\gamma = \Delta$.

7) Finally, suppose that $\alpha = \gamma = \Delta$. In this case $\tau = \delta$ and the desired inequality $S \le \tau$ is obvious. The proof is completed. □

Appendix

Here we describe the behaviour of solutions of SDEs. We use the notations $\mathcal{F}$, $\mathcal{F}_t$, X, and ζ introduced in Subsection 5.2.

Let us consider SDE (5.1) and assume that conditions (5.2) and (5.3) are satisfied. According to Proposition 5.6, this equation has a unique solution P in the sense of Definition 5.4. Consider the sets

$$D = \bigl\{\zeta = \infty,\ \overline{\lim_{t\to\infty}} X_t = \infty,\ \underline{\lim_{t\to\infty}} X_t = -\infty\bigr\},$$
$$B_+ = \bigl\{\zeta = \infty,\ \lim_{t\to\infty} X_t = \infty\bigr\},$$
$$C_+ = \bigl\{\zeta < \infty,\ \lim_{t\uparrow\zeta} X_t = \infty\bigr\},$$
$$B_- = \bigl\{\zeta = \infty,\ \lim_{t\to\infty} X_t = -\infty\bigr\},$$
$$C_- = \bigl\{\zeta < \infty,\ \lim_{t\uparrow\zeta} X_t = -\infty\bigr\}.$$

Define ρ, s, $s(\infty)$, $s(-\infty)$ by formulas (5.6)–(5.9).

The statements below follow from [4; Ch. 4].

Proposition A.1. *Either* $\mathsf{P}(D) = 1$ *or* $\mathsf{P}(B_+ \cup B_- \cup C_+ \cup C_-) = 1$.

Proposition A.2. **(i)** *If* $s(\infty) = \infty$, *then* $\mathsf{P}(B_+) = \mathsf{P}(C_+) = 0$.
(ii) *If* $s(\infty) < \infty$ *and* $(s(\infty) - s)/\rho\sigma^2 \notin L^1_{\mathrm{loc}}(\infty)$, *then* $\mathsf{P}(B_+) > 0$, $\mathsf{P}(C_+) = 0$.
(iii) *If* $s(\infty) < \infty$ *and* $(s(\infty) - s)/\rho\sigma^2 \in L^1_{\mathrm{loc}}(\infty)$, *then* $\mathsf{P}(B_+) = 0$, $\mathsf{P}(C_+) > 0$.

Clearly, Proposition A.2 has its analog for the behaviour at $-\infty$.

For any $a, c \in \mathbb{R}$, set $T_a = \inf\{t \in [0,\infty) \colon X_t = a\}$ (here $\inf\emptyset = \infty$) and set $T_{a,c} = T_a \wedge T_c$.

Proposition A.3. **(i)** *For any* $a \in \mathbb{R}$, $\mathsf{P}(T_a < \infty) > 0$.
(ii) *Let* $a \in (-\infty, x_0)$. *Then* $T_a < \infty$ P-*a.s.* $\iff s(\infty) = \infty$.
(iii) *Let* $a \in (x_0, \infty)$. *Then* $T_a < \infty$ P-*a.s.* $\iff s(-\infty) = -\infty$.
(iv) *Let* $a \in (-\infty, x_0)$, $c \in (x_0, \infty)$. *Then* $T_{a,c} < \infty$ P-*a.s. Moreover,* $\mathsf{P}(T_a < T_c) > 0$ *and* $\mathsf{P}(T_c < T_a) > 0$.

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