FAMILIES OF CONSISTENT PROBABILITY MEASURES

A.S. Cherny

Moscow State University,
Faculty of Mechanics and Mathematics,
Department of Probability Theory,
119992 Moscow, Russia.
E-mail: cherny@mech.math.msu.su

Abstract. This paper deals with the following problem. Suppose that $(\mathsf{P}_t)_{t\ge 0}$ is a family of consistent probability measures defined on a filtration $(\mathcal{F}_t)_{t\ge 0}$. Does there exist a measure P on the σ-field $\bigvee_{t\ge 0}\mathcal{F}_t$ such that $\mathsf{P}|\mathcal{F}_t=\mathsf{P}_t$? The answer is positive for the spaces $C(\mathbb{R}_+,\mathbb{R}^d)$ and $D(\mathbb{R}_+,\mathbb{R}^d)$ endowed with the natural filtration. We prove this statement using a simple method based on the *Prokhorov criterion* of weak compactness.

Key words and phrases. Consistent probability measures, extension of measures, Skorokhod space, Prokhorov criterion.

1 Introduction

Let $(\Omega,(\mathcal{F}_t)_{t\ge 0})$ be a filtered space. Suppose that, for every $t\ge 0$, P_t is a probability measure on $\mathcal{F}_t$ and these measures are *consistent*, i.e. $\mathsf{P}_t|\mathcal{F}_s=\mathsf{P}_s$ for $s\le t$ ($\mathsf{P}_t|\mathcal{F}_s$ denotes the restriction of P_t to $\mathcal{F}_s$). The question arises whether there exists a measure P on $\bigvee_{t\ge 0}\mathcal{F}_t$ such that $\mathsf{P}|\mathcal{F}_t=\mathsf{P}_t$ for each $t\ge 0$.

In the general case the answer to this question is negative (see Section 4).

However, in many natural cases the answer is positive. The following statement is proved in [3, Ch. V, §4].

Proposition 1. *Let $(\Omega,(\mathcal{F}_n)_{n=1}^\infty)$ be a filtered space. Suppose that*

i) *for each n, there exists a Polish space X_n such that $\mathcal{F}_n$ is isomorphic to the Borel σ-field $\mathcal{B}(X_n)$;*

ii) *for any sequence $(A_n)_{n=1}^\infty$ such that A_n is an atom of $\mathcal{F}_n$ ($n\in\mathbb{N}$), one has: $\bigcap_n A_n\ne\varnothing$.*

Then any sequence $(\mathsf{P}_n)_{n=1}^\infty$ of consistent probability measures on $(\mathcal{F}_n)$ can be extended to a measure on $\bigvee_n\mathcal{F}_n$.

The problem under consideration usually arises for the path spaces $C(\mathbb{R}_+,\mathbb{R}^d)$ and $D(\mathbb{R}_+,\mathbb{R}^d)$ rather than for the abstract measurable spaces. The following statement can easily be derived from Proposition 1.

Theorem 1. *Let $C(\mathbb{R}_+,\mathbb{R}^d)$ (respectively, $D(\mathbb{R}_+,\mathbb{R}^d)$) be the space of continuous (respectively, càdlàg) functions $\mathbb{R}_+\to\mathbb{R}^d$. Let $(X_t)_{t\ge 0}$ be the coordinate process and $\mathcal{F}_t=\sigma(X_s;\,s\le t)$ be the natural filtration. Suppose that (P_t) is a family of consistent probability measures on $(\mathcal{F}_t)$. Then there exists a unique measure P on $\mathcal{F}=\sigma(X_s;\,s\ge 0)$ such that $\mathsf{P}|\mathcal{F}_t=\mathsf{P}_t$ for any $t\ge 0$.*

The proof of this statement for the space $C(\mathbb{R}_+, \mathbb{R}^d)$ can also be found in [5, (1.3.5)].

The proofs in [3] and [5] are based on rather complicated arguments from the measure theory. In this paper (Section 2), we give a simple proof of Theorem 1 based on the *Prokhorov criterion* of weak compactness. In Section 3, we present the application of the obtained results. Section 4 contains an example which shows that, for an abstract probability space endowed with a filtration $(\mathcal{F}_t)$, a family of consistent probability measures on $(\mathcal{F}_t)$ may not be extended to a measure on $\bigvee_{t\geq 0} \mathcal{F}_t$.

2 The Proof of Theorem 1

Proof. We will prove the statement for the space $D(\mathbb{R}_+, \mathbb{R}^d)$. For each $n \in \mathbb{N}$, the map

$$F_n : D(\mathbb{R}_+, \mathbb{R}^d) \ni X \longmapsto Y \in D(\mathbb{R}_+, \mathbb{R}^d)$$

defined as $Y_t = X_{t\wedge n}$ is $\mathcal{F}_n|\mathcal{F}$-measurable. Set $\mathsf{Q}_n = \mathsf{P}_n \circ F_n^{-1}$ (i.e. Q_n is a measure on $\mathcal{F}$). Note that $\mathcal{F}$ coincides with the Borel σ-field $\mathcal{B}(D(\mathbb{R}_+, \mathbb{R}^d))$.

We will prove that the sequence $(\mathsf{Q}_n)_{n=1}^\infty$ is *tight*. The necessary and sufficient condition of tightness is as follows (see [2, Ch. VI, §3b]): *for any $N \in \mathbb{N}$, $\varepsilon > 0$ and $\eta > 0$, there exist $n_0 \in \mathbb{N}$, $K > 0$ and $\theta > 0$ such that*

$$n \geq n_0 \implies \mathsf{Q}_n\bigl(\sup_{t\leq N} |X_t| < K\bigr) > 1 - \varepsilon, \tag{1}$$

$$n \geq n_0 \implies \mathsf{Q}_n\bigl(\omega'_N(X, \theta) < \eta\bigr) > 1 - \varepsilon, \tag{2}$$

where $\omega'_N(X, \theta)$ is the following modulus of "continuity":

$$\omega'_N(X, \theta) = \inf\bigl\{\max_{i\leq m} \sup_{t_{i-1}\leq s\leq r<t_i} |X_r - X_s| :$$
$$0 = t_0 < \ldots < t_m = N,\ \inf_{i<m}(t_i - t_{i-1}) \geq \theta\bigr\}.$$

Let us fix $N \in \mathbb{N}$, $\varepsilon > 0$ and $\eta > 0$. The one-point set $\{\mathsf{Q}_N\}$ is tight. Hence, there exist $K > 0$ and $\theta > 0$ such that

$$\mathsf{Q}_N\bigl(\sup_{t\leq N} |X_t| < K\bigr) > 1 - \varepsilon, \tag{3}$$

$$\mathsf{Q}_N\bigl(\omega'_N(X, \theta) < \eta\bigr) > 1 - \varepsilon. \tag{4}$$

It follows from the condition $\mathsf{P}_n|\mathcal{F}_N = \mathsf{P}_N$ $(n \geq N)$ that $\mathsf{Q}_n|\mathcal{F}_N = \mathsf{Q}_N|\mathcal{F}_N$ $(n \geq N)$. Therefore, (3), (4) yield (1), (2) with $n_0 = N$.

Thus, the sequence $(\mathsf{Q}_n)_{n=1}^\infty$ is tight. The *Prokhorov criterion* implies that there exist a sequence $n_k \to \infty$ and a measure P on $\mathcal{F}$ such that Q_{n_k} converges weakly to P as $k \to \infty$. We state that $\mathsf{P}|\mathcal{F}_t = \mathsf{P}_t$ for all $t \geq 0$. In order to prove this assertion, let us fix $M \in \mathbb{N}$ and consider the map

$$G : D(\mathbb{R}_+, \mathbb{R}^d) \ni X \longmapsto Y \in D([0, M], \mathbb{R}^d)$$

defined as $Y_t = X_t$ $(t \leq M)$. Let $\widetilde{k}$ denote the minimal k satisfying the inequality $n_k \geq M$. Set

$$\mathsf{R} = \mathsf{P} \circ G^{-1}, \quad \mathsf{R}_k = \mathsf{Q}_{n_k} \circ G^{-1} \quad (k \geq \widetilde{k}).$$

Obviously, $\mathsf{R}_k = \mathsf{R}_{\widetilde{k}}$ for $k \geq \widetilde{k}$. The map G is continuous when $D(\mathbb{R}_+, \mathbb{R}^d)$ and $D([0, M], \mathbb{R}^d)$ are endowed with the *Skorokhod topology*. Therefore, R_k converges weakly to R as $k \to \infty$. Since $\mathsf{R}_k = \mathsf{R}_{\widetilde{k}}$ for $k \geq \widetilde{k}$, we conclude that $\mathsf{R} = \mathsf{R}_{\widetilde{k}}$. Hence, the measures P and $\mathsf{Q}_{n_{\widetilde{k}}}$ coincide on the σ-field $G^{-1}(\mathcal{B}(D([0, M], \mathbb{R}^d))) = \mathcal{F}_M$. Due to the inequality $n_{\widetilde{k}} \geq M$, we have

$$\mathsf{P}|\mathcal{F}_M = \mathsf{Q}_{n_{\widetilde{k}}}|\mathcal{F}_M = \mathsf{Q}_M|\mathcal{F}_M = \mathsf{P}_M.$$

As M is arbitrary, we deduce that $\mathsf{P}|\mathcal{F}_t = \mathsf{P}_t$ for each $t \ge 0$.

The uniqueness of P follows from the equality $\mathcal{F} = \bigvee_{t\ge 0} \mathcal{F}_t$.

The proof of the theorem for the space $C(\mathbb{R}_+, \mathbb{R}^d)$ is the same except that in formulas (1)–(4) one should replace $\omega'_N(X,\theta)$ by the following modulus of continuity:

$$\omega_N(X,\theta) = \sup\{|X_r - X_s| : 0 \le s \le r \le N,\ |r-s| \le \theta\}.$$

Remark. Theorem 1 remains valid if the filtration $(\mathcal{F}_t)$ is replaced by its right modification $\mathcal{G}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$. In order to prove this statement, we set $\widetilde{\mathsf{P}}_t = \mathsf{P}_t|\mathcal{F}_t$, use Theorem 1 to construct a measure P on $\bigvee_{t\ge 0} \mathcal{F}_t$ such that $\mathsf{P}|\mathcal{F}_t = \mathsf{P}_t$ $(t \ge 0)$ and then apply the obvious equalities:

$$\mathsf{P}|\mathcal{G}_t = (\mathsf{P}|\mathcal{F}_{t+1})|\mathcal{G}_t = \widetilde{\mathsf{P}}_{t+1}|\mathcal{G}_t = (\mathsf{P}_{t+1}|\mathcal{F}_{t+1})|\mathcal{G}_t = \mathsf{P}_{t+1}|\mathcal{G}_t = \mathsf{P}_t.$$

3 The Application of the Obtained Results

In this section, we will consider the following problem.

Let $(B_t;\, t \ge 0)$ be a Brownian motion on a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$ and τ be a $(\mathcal{F}_t)$-stopping time. We will present a necessary and sufficient condition for the process

$$M_t = \exp\Big\{B_{t\wedge\tau} - \frac{1}{2}\, t \wedge \tau\Big\}, \quad t \ge 0$$

to be a uniformly integrable martingale.

Obviously, M is uniformly integrable if and only if

$$\mathsf{E}_{\mathsf{P}} \exp\Big\{B_\tau - \frac{1}{2}\tau\Big\} = 1, \tag{5}$$

where

$$\exp\Big\{B_\tau - \frac{1}{2}\tau\Big\} = 0 \tag{6}$$

on the set $\{\tau = \infty\}$. Let us consider the two-dimensional process $(X_t;\, t \ge 0)$ of the form:

$$X_t^1 = B_t, \quad X_t^2 = (t-\tau)^+.$$

Then (5) is equivalent to:

$$\mathsf{E}_{\mathsf{Q}} \exp\Big\{X_T^1 - \frac{1}{2}T\Big\} = 1, \tag{7}$$

where Q is the distribution of the process X (Q is a measure on $C(\mathbb{R}_+, \mathbb{R}^2)$) and $T = \inf\{t \ge 0 :\, X_t^2 > 0\}$.

In order to investigate (7), we set $\mathcal{G}_t = \bigcap_{\varepsilon>0} \sigma(X_s;\, s \le t+\varepsilon)$, $\mathsf{Q}_t = \mathsf{Q}|\mathcal{G}_t$ and consider the measures $(\widetilde{\mathsf{Q}}_t)_{t\ge 0}$ defined by

$$\frac{d\widetilde{\mathsf{Q}}_t}{d\mathsf{Q}_t} = \exp\Big\{X_t^1 - \frac{1}{2}t\Big\}.$$

Obviously, $(\widetilde{\mathsf{Q}}_t)_{t\ge 0}$ is a consistent family of probability measures. By the Remark following the proof of Theorem 1, there exists a measure $\widetilde{\mathsf{Q}}$ on $\mathcal{F} = \sigma(X_s;\, s \ge 0)$ such that $\widetilde{\mathsf{Q}}|\mathcal{G}_t = \widetilde{\mathsf{Q}}_t$ $(t \ge 0)$. Now, set $\mathsf{Q}_T = \mathsf{Q}|\mathcal{G}_T$, $\widetilde{\mathsf{Q}}_T = \widetilde{\mathsf{Q}}|\mathcal{G}_T$. By [2, Ch. III, §3a],

$$\frac{d\widetilde{\mathsf{Q}}_T|\{T<\infty\}}{d\mathsf{Q}_T|\{T<\infty\}} = \exp\Big\{X_T^1 - \frac{1}{2}T\Big\}.$$

Therefore,

$$\widetilde{\mathsf{Q}}\{T<\infty\} = \mathsf{E}_{\mathsf{Q}}\Big[I(T<\infty)\,\exp\Big\{X_T^1 - \frac{1}{2}T\Big\}\Big] = \mathsf{E}_{\mathsf{Q}}\exp\Big\{X_T^1 - \frac{1}{2}T\Big\}.$$

Here, we use the fact that $\exp\{X_T^1 - \frac{1}{2}T\} = 0$ on the set $\{T=\infty\}$ (see (6)). Thus, (7) is equivalent to the following condition:

$$\widetilde{\mathsf{Q}}\{T<\infty\} = 1. \tag{8}$$

It is usually hard to verify condition (8), except for some special cases. For example, suppose that $\tau = \inf\{t \ge 0 : B_t \ge \varphi(t)\}$, where $\varphi \in C(\mathbb{R}_+)$, $\varphi(0) > 0$. In that case the distribution of the process $(X_t^1;\, t \ge 0)$ under $\widetilde{\mathsf{Q}}$ coincides with the distribution of the Brownian motion with the unit drift. Therefore, $\widetilde{\mathsf{Q}}\{T<\infty\} = 1$ if and only if the function $\varphi(t) - t$ is a *lower function* of Brownian motion.

4 An Example

Theorem 1 may not be valid if $(\mathcal{F}_t)$ is replaced by another filtration. The following example is taken from [1].

Example 1. Let $(\mathcal{G}_t)_{t\in[0,1]}$ be the filtration on $C([0,1])$ defined as $\mathcal{G}_t = \sigma(X_s;\, s \le t) \vee \sigma(X_1)$, where $X = (X_t)_{t\in[0,1]}$ is the coordinate process. Let P be the Wiener measure on $\mathcal{G} = \mathcal{G}_1$. It is well known that under P, the coordinate process admits the representation

$$X_t = \int_0^t \frac{X_1 - X_s}{1-s}\, ds + B_t = \int_0^t \alpha_s\, ds + B_t,$$

where B is $(\mathcal{G}_t)$-Brownian motion. Let us consider the family $(\mathsf{Q}_t)_{t\in[0,1)}$ defined as

$$\frac{d\mathsf{Q}_t}{d\mathsf{P}_t} = \exp\Big\{-\int_0^t \alpha_s\, dX_s + \frac{1}{2}\int_0^t \alpha_s^2\, ds\Big\},$$

where $\mathsf{P}_t = \mathsf{P}|\mathcal{G}_t$. Then $(\mathsf{Q}_t)_{t\in[0,1)}$ is a family of consistent probability measures on $(\mathcal{G}_t)_{t\in[0,1)}$.

Suppose that there exists a measure Q on $\mathcal{G}$ such that $\mathsf{Q}|\mathcal{G}_t = \mathsf{Q}_t$ for all $t \in [0,1)$. By Girsanov's theorem,

$$\mathrm{Law}(X_s;\, s \le t|\mathsf{Q}_t) = \mathrm{Law}(B_s;\, s \le t) = \mathrm{Law}(X_s;\, s \le t|\mathsf{P}_t)$$

for each $t < 1$. Hence,

$$\mathrm{Law}(X_s;\, s \le 1|\mathsf{Q}) = \mathrm{Law}(X_s;\, s \le 1|\mathsf{P}).$$

Therefore,

$$\mathsf{Q}|\sigma(X_s;\, s \le 1) = \mathsf{P}|\sigma(X_s;\, s \le 1).$$

As $\sigma(X_s;\, s \le 1) = \mathcal{G}$, we conclude that $\mathsf{Q} = \mathsf{P}$. This leads to a contradiction since

$$\mathsf{Q}|\mathcal{G}_t = \mathsf{Q}_t \ne \mathsf{P}_t = \mathsf{P}|\mathcal{G}_t.$$

Thus, the family $(\mathsf{Q}_t)_{t\in[0,1)}$ can not be extended to the σ-field $\mathcal{G}$.

Remark. The book [4, Ch. II, §3] contains a different example. It shows that, for a general filtered space, the family of consistent probability measures cannot be extended to $\bigvee_{t\ge 0} \mathcal{F}_t$.

Acknowledgement. I am grateful to professor A.N. Shiryaev for useful discussions and recommendations. I express my thanks to professor H.-J. Engelbert for his important comments and suggestions.

References

[1] *Föllmer H., Imkeller P.* Anticipation cancelled by a Girsanov transformation: a paradox on Wiener space. — Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, V. 29 (1993), No. 4, p. 569–586.

[2] *Jacod J., Shiryaev A.N.* Limit theorems for random processes. Springer-Verlag, 1987.

[3] *Parthasarathy K.R.* Probability measures on metric spaces. Academic Press, New York, London, 1967.

[4] *Shiryaev A.N.* Probability. Springer-Verlag, 1996.

[5] *Stroock D.W., Varadhan S.R.S.* Multidimensional diffusion processes. Springer-Verlag, 1979.