GENERAL ARBITRAGE PRICING MODEL: TRANSACTION COSTS

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Abstract. In this paper we apply the general framework introduced in [2] to two models with transaction costs:

- a dynamic model with an infinite number of assets;
- a model with European call options as basic assets.

In particular, it is proved that a dynamic model with an infinite number of assets satisfies the No Generalized Arbitrage condition (this notion was introduced in [2]) if and only if there exist an equivalent measure and a martingale with respect to this measure that lies (componentwise) between the discounted ask and bid price processes. Furthermore, the set of fair prices of a contingent claim coincides with the set of expectations of the payoff with respect to these measures.

Our approach to arbitrage pricing in models with transaction costs differs from the existing ones.

Key words and phrases. Delta-martingale, fair price, fundamental theorem of asset pricing, general arbitrage pricing model, generalized arbitrage, risk-neutral measure, set of attainable incomes, transaction costs.

1 Introduction

1. Purpose of the paper. Models with transaction costs have recently attracted much attention in the financial mathematics literature. Let us mention, in particular, the papers [4], [6], [12], [13], [14], [15], [16], [17], [19], [22] dealing with arbitrage pricing in such models. These papers differ in the level of generality, conditions imposed on price processes, definition of a strategy, definition of a price, and the form of representation of results.

In the paper [2], we introduced a unified approach to pricing contingent claims through a new concept of *generalized arbitrage*. (The necessary definitions and statements from [2] are collected in Section 2.) In the framework of a general arbitrage pricing model, we proved in [2] the fundamental theorem of asset pricing and established the form of the fair price intervals. The general approach of [2] allows one to consider in a simple and unified manner various models of arbitrage pricing theory, some of which have so far been investigated separately and by different techniques. These include

- static as well as dynamic models;

- models with an infinite number of assets;
- models with transaction costs.

The purpose of this paper is to "project" the general results of [2] on two models with transaction costs.

2. Dynamic model with an infinite number of assets. This model is considered in Section 3. In order to apply the general results of [2], one only needs to establish the structure of the set of equivalent *risk-neutral measures* (see Definition 2.3). We prove that an equivalent measure Q is a risk-neutral measure if and only if there exists a Q-martingale that lies componentwise between the discounted ask and bid price processes. Then the general results of [2] show that the absence of generalized arbitrage (see Definition 2.2) is equivalent to the existence of such a measure, while the set of fair prices of a contingent claim coincides with the set of expectations of its payoff with respect to the class of these measures.

Our approach to arbitrage pricing in dynamic models with transaction costs is different from the approaches of all the papers mentioned above. First of all, our model is completely general in the sense that we consider an arbitrary Ω, the continuous-time case (so that the discrete-time case is covered as well), and arbitrary (not only proportional) transaction costs. There are no assumptions on the probability structure of price evolution (like the assumption that the price is a geometric Brownian motion). We consider a model with an arbitrary number of assets, while all the above mentioned papers consider only a finite number of assets. An important conceptual difference between our model and the majority of models mentioned above is as follows. In most of the above mentioned papers a contingent claim is modeled as a multidimensional vector (its i-th component means the amount of the assets of type i obtained by a holder of the claim). In contrast, here we use the monetary representation, i.e. we consider a contingent claim as a one-dimensional random variable. Another important distinctive feature of our approach is that the price of a contingent claim is defined not through sub- and superreplication, but directly through the No Generalized Arbitrage condition (see Definition 2.7).

In most aspects mentioned above, our model is similar to the model of Jouini and Kallal [12], but there is a number of essential differences between the two models. The approach of Jouini and Kallal might be considered as a "transaction costs' extension" of the approach of Harrison and Kreps [8], while our model is the "transaction costs' projection" of the general arbitrage pricing model introduced in [2]. The most important difference between our approach and the approaches of [8] and [12] is that these papers employ the L^2-setting (in particular, the price processes and the capital processes are assumed to be square-integrable and the densities $d\mathsf{Q}/d\mathsf{P}$ or risk-neutral measures should also be square-integrable), while we employ the L^0-setting.

We also study in our framework the convergence of the fair price intervals of a European call option $(S_T - K)^+$ in the Black–Scholes model with proportional transaction costs as the coefficient of transaction costs tends to zero. It is shown that the fair price interval tends to the trivial one, i.e. to $((S_0 - K)^+, S_0)$. Although our framework differs from the existing ones, this result agrees with the results of [5], [20], and [24], where the same problem was considered. The financial interpretation is as follows: in the model under consideration, the fair price interval obtained by dynamic hedging coincides with the fair price interval obtained by static hedging.

3. Model with European call options as basic assets. This model is considered in Section 4 (it is again a particular case of the general arbitrage pricing model). We provide a simple geometric representation of the class of risk-neutral measures. The frictionless variant of this model is very popular in financial mathematics (see, in particular, [1], [10]) and was analyzed in [2; Sect. 6] within our general framework. The main idea of considering such models is that taking into account the market prices of traded derivatives enables one to narrow considerably fair price intervals.

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2 Generalized Arbitrage

Here we recall some basic definitions and facts from [2].

Definition 2.1. A *general arbitrage pricing model* is a quadruple $(\Omega, \mathcal{F}, \mathsf{P}, A)$, where $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space and A is a convex cone in the space of all random variables.

From the financial point of view, A is the set of all discounted incomes that can be obtained by trading a certain amount of assets. In the frictionless models, A is a linear space. In the models with transaction costs, A is a cone.

`Notation`. `(i)` Set

$$B = \Big\{ Z \in L^0 : \text{ there exist } (X_n)_{n\in\mathbb{N}} \in A \text{ and } a \in \mathbb{R} \\ \text{such that } X_n \ge a \ \mathsf{P}\text{-a.s. and } Z = \lim_{n\to\infty} X_n \ \mathsf{P}\text{-a.s.} \Big\}. \tag{2.1}$$

`(ii)` For $Z \in B$, denote $\gamma(Z) = 1 - \operatorname{essinf}_{\omega\in\Omega} Z(\omega)$ and set

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L^0_+\}, \\ A_2(Z) &= \left\{ \frac{X}{Z + \gamma(Z)} : X \in A_1 \right\}, \\ A_3(Z) &= A_2(Z) \cap L^\infty, \\ A_4(Z) &= \text{closure of } A_3(Z) \text{ in } \sigma(L^\infty, L^1(\mathsf{P})). \end{aligned} \tag{2.2}$$

Here L^0_+ is the set of $\mathbb{R}_+$-valued elements of L^0; L^∞ is the space of bounded elements of L^0; $\sigma(L^\infty, L^1(\mathsf{P}))$ denotes the weak topology on L^∞ induced by the space $L^1(\mathsf{P})$ of the P-integrable random variables on $(\Omega, \mathcal{F}, \mathsf{P})$.

Definition 2.2. A model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the *No Generalized Arbitrage* (NGA) condition if for any $Z \in B$, we have $A_4(Z) \cap L^0_+ = \{0\}$.

Definition 2.3. An equivalent *risk-neutral measure* is a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X^- \ge \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$ (we use the notation $X^- = (-X) \vee 0$, $X^+ = X \vee 0$). The expectations $\mathsf{E}_\mathsf{Q} X^-$ and $\mathsf{E}_\mathsf{Q} X^+$ here may take on the value $+\infty$. The set of equivalent risk-neutral measures will be denoted by $\mathcal{R}$.

`Notation`. For $Z \in B$, we will denote by $\mathcal{R}(Z)$ the set of the probability measures $\mathsf{Q} \sim \mathsf{P}$ with the property: for any $X \in A$ such that $X \ge -\alpha Z - \beta$ P-a.s. with some $\alpha, \beta \in \mathbb{R}_+$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \le 0$.

Lemma 2.4. *For any $Z \in B$, we have $\mathcal{R} \subseteq \mathcal{R}(Z)$.*

Assumption 2.5. There exists $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$ (in particular, both sets might be empty).

Theorem 2.6 (Fundamental theorem of asset pricing). *Suppose that Assumption 2.5 is satisfied. Then the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition if and only if there exists an equivalent risk-neutral measure.*

Now, let F be a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ meaning the discounted payoff of a contingent claim.

Definition 2.7. A real number x is a *fair price* of F if the extended model $(\Omega, \mathcal{F}, \mathsf{P}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGA condition. The set of fair prices of F will be denoted by $I(F)$.

Theorem 2.8 (Pricing contingent claims). *Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies Assumption 2.5 and the NGA condition, while F is bounded below. Then*

$$I(F) = \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\}.$$

The expectation $\mathsf{E}_{\mathsf{Q}} F$ here is taken in the sense of finite expectations, i.e. we consider only those Q, for which $\mathsf{E}_{\mathsf{Q}} F < \infty$.

Let us illustrate the setup introduced above by a static model with a finite number of assets.

Example 2.9. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let $S_0^a, S_0^b \in \mathbb{R}^d$ and S_1^a, S_1^b be $\mathbb{R}^d$-valued random vectors. From the financial point of view, S_n^{ai} (resp., S_n^{bi}) is the discounted ask (resp., bid) price of the i-th asset at time n (so that $S_n^a \geq S_n^b$ componentwise). Define the set of attainable incomes by

$$A = \left\{ \sum_{i=1}^{d} [g^i (S_1^{bi} - S_0^{ai}) + h^i (-S_1^{ai} + S_0^{bi})] : g^i, h^i \in \mathbb{R}_+ \right\}.$$

Then the NGA condition is equivalent to the traditional No Arbitrage (NA) condition defined as: $A \cap L_+^0 = \{0\}$. (Consequently, the set of fair prices would remain unchanged if we replaced the NGA condition in the definition of a fair price by the NA condition.) Indeed, the implication NGA$\Rightarrow$NA is obvious, while the implication NA$\Rightarrow$NGA is proved as follows. Assume the NA condition and consider the measure $\mathsf{P}' = c(\|S_1^a\| \vee \|S_1^b\| \vee 1)^{-1} \mathsf{P}$, where c is the normalizing constant. By the Kreps–Yan theorem (see [18] or [25]), there exists a probability measure $\mathsf{Q} \sim \mathsf{P}'$ such that the density $d\mathsf{Q}/d\mathsf{P}'$ is bounded and $\mathsf{E}_{\mathsf{Q}} X \leq 0$ for any $X \in A$. Then $\mathsf{Q} \in \mathcal{R}$ and, by Theorem 2.6, the NGA is satisfied (note the proof of the implication $\mathcal{R} \neq \emptyset \;\Rightarrow\;$ NGA in this theorem does not employ Assumption 2.5). $\square$

3 Dynamic Model with Infinite Number of Assets

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial and $(\mathcal{F}_t)$ is right-continuous. Let $(S_t^{ai})_{t\in[0,T]}$ and $(S_t^{bi})_{t\in[0,T]}$, $i \in I$ be a family of real-valued $(\mathcal{F}_t)$-adapted càdlàg processes. From the financial point of view, S_t^{ai} (resp., S_t^{bi}) is the discounted ask (resp., bid) price of the i-th asset at time t (so that $S_t^a \geq S_t^b$ componentwise). Define the set of attainable incomes by

$$A = \biggl\{\sum_{n=0}^{N}\sum_{i\in I}\bigl[-H_n^i I(H_n^i > 0)S_{u_n}^{ai} - H_n^i I(H_n^i < 0)S_{u_n}^{bi}\bigr] : \\ N \in \mathbb{N},\ u_0 \leq \dots \leq u_N \text{ are } (\mathcal{F}_t)\text{-stopping times, } H_n^i \text{ is } \mathcal{F}_{u_n}\text{-measurable,} \\ H_n^i = 0 \text{ for all } i, \text{ except for a finite set, and } \sum_{n=0}^{N} H_n^i = 0 \text{ for any } i\biggr\}.$$

Here H_n^i means the amount of the i-th asset that is bought at time u_n (so that $\sum_{k=0}^n H_k^i$ is the total amount of the i-th asset held at time u_n).

`Remark`. Consider a model with no transaction costs (i.e. $S^a = S^b = S$). Then for any i and any H_n^i such that $\sum_{n=0}^N H_n^i = 0$, we can write

$$\sum_{n=0}^{N}[-H_n^i S_{u_n}^i] = \sum_{n=1}^{N}\biggl(\sum_{k=0}^{n-1} H_k^i\biggr)(S_{u_n}^i - S_{u_{n-1}}^i).$$

Thus, in this model the set A admits a simpler description:

$$A = \biggl\{\sum_{n=1}^{N}\sum_{i\in I} H_n^i(S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \leq \dots \leq u_N \text{ are } (\mathcal{F}_t)\text{-stopping times,} \\ H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H_n^i = 0 \text{ for all } i, \text{ except for a finite set}\biggr\}.$$

We will assume that each process S^{bi} is positive. We will also suppose that, for any $i \in I$, there exists a constant $\gamma^i > 0$ such that $S^{ai} \leq \gamma^i S^{bi}$. Finally, we assume that, for any $t \in [0,T]$, there exists $Y_t \in B$ (B is defined by (2.1)) with the property: for any $i \in I$, there exist $\alpha, \beta > 0$ such that $S_t^{bi} \leq \alpha Y_t + \beta$ a.s. This assumption is automatically satisfied in natural models.

Indeed, if I is finite, then the above assumption is satisfied with

$$Y_t = \sum_{i\in I}(S_t^{bi} - S_0^{ai}).$$

If I is countable, then the above assumption is satisfied with

$$Y_t = \sum_{i\in I}\lambda^i(S_t^{bi} - S_0^{ai}),$$

where constants $\lambda^i > 0$ are chosen in such a way that $\sum_{i\in I}\lambda^i S_t^{bi} < \infty$ a.s. and $\sum_{i\in I}\lambda^i S_0^{ai} < \infty$.

If S^{bi} is the discounted bid price process of a zero-coupon bond with maturity i, then S^{bi} takes on values in $[0,1]$, and the above assumption is satisfied with $Y_t = 0$.

In order to get the fundamental theorem of asset pricing and to obtain the form of the fair price intervals, it is sufficient to prove that Assumption 2.5 is satisfied and to find the structure of risk-neutral measures. We call the corresponding statement the *Key Lemma* of the section.

`Notation`. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \text{for each } i \in I, \text{there exists an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale } M^i \\ \text{such that, for any } t \in [0,T],\ S_t^{bi} \le M_t^i \le S_t^{ai}\ \ \mathsf{Q}\text{-a.s.}\}.$$

Here M^i need not be càdlàg.

Key Lemma 3.1. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}\biggl(\sum_{t \in \mathbb{Q} \cap [0,T]} \lambda_t Y_t\biggr) = \mathcal{M},$$

where constants $\lambda_t > 0$ *are chosen in such a way that* $\sum_{t \in \mathbb{Q}\cap[0,T]} \lambda_t |Y_t| < \infty$ *a.s. and* $\sum_{t \in \mathbb{Q}\cap[0,T]} \lambda_t \operatorname{essinf}_{\omega \in \Omega} Y_t(\omega) < \infty$.

The proof employs two auxiliary statements. The first of them was proved by Jouini and Kallal [12; Lem. 3] (see Choulli and Stricker [3] for a related result). Actually, Jouini and Kallal employ an additional assumption that X and Y are càdlàg, but a slight modification of their proof allows one to get rid of this assumption.

Lemma 3.2. *Let* X *be a supermartingale and* Y *be a submartingale on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathsf{P})$ *with a right-continuous filtration* (X *and* Y *are not necessarily càdlàg). Suppose that, for any* $t \in [0,T]$, $X_t \le Y_t$ *a.s. Then there exists an* $(\mathcal{F}_t)$*-martingale* M *such that, for any* $t \in [0,T]$, $X_t \le M_t \le Y_t$ *a.s.*

We will also employ the following statement (see [11] or [23; Ch. II, § 1c]):

Lemma 3.3. *Let* $(X_n)_{n=0,\dots,N}$ *be an* $(\mathcal{F}_n)$*-local martingale such that* $\mathsf{E}|X_0| < \infty$ *and* $\mathsf{E}X_N^- < \infty$. *Then* X *is an* $(\mathcal{F}_n)$*-martingale.*

`Proof of Key Lemma 3.1`. Denote $\sum_{t \in \mathbb{Q}\cap[0,T]} \lambda_t Y_t$ by Z_0.

Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 2.4.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(Z_0)$. Fix $i \in I$ and $(\mathcal{F}_t)$-stopping times $u \le v$. Let us prove that

$$\mathsf{E}_{\mathsf{Q}}(S_v^{bi} \mid \mathcal{F}_u) \le S_u^{ai}. \tag{3.1}$$

For $n \in \mathbb{N}$, set

$$u_n = \sum_{k=1}^n \frac{kT}{n} I\biggl(\frac{(k-1)T}{n} < u \le \frac{kT}{n}\biggr),$$
$$v_n = \sum_{k=1}^n \frac{kT}{n} I\biggl(\frac{(k-1)T}{n} < v \le \frac{kT}{n}\biggr).$$

Then, for any $n \le m$ and any $D \in \mathcal{F}_{u_m}$ such that $S^{ai}_{u_m}$ is bounded on D, we have $u_m \le v_n$ and

$$\mathsf{E}_{\mathsf{Q}} I_D(S^{bi}_{v_n} - S^{ai}_{u_m}) \le 0,$$

which implies that

$$\mathsf{E}_{\mathsf{Q}}(S^{bi}_{v_n} \mid \mathcal{F}_{u_m}) \le S^{ai}_{u_m}. \tag{3.2}$$

As u_m decreases to u pointwise, we have $\mathcal{F}_{u_m} \subseteq \mathcal{F}_{u_{m-1}}$ and $\bigcap_{m=1}^{\infty} \mathcal{F}_{u_m} = \mathcal{F}_u$ (see [21; Ch. I, Ex. 4.17]). Therefore,

$$\mathsf{E}_{\mathsf{Q}}(S^{bi}_{v_n} \mid \mathcal{F}_{u_m}) \xrightarrow[m\to\infty]{\mathsf{Q}\text{-a.s.}} \mathsf{E}_{\mathsf{Q}}(S^{bi}_{v_n} \mid \mathcal{F}_u)$$

(see [21; Ch. II, Cor. 2.4]) and (3.2) yields

$$\mathsf{E}_{\mathsf{Q}}(S^{bi}_{v_n} \mid \mathcal{F}_u) \le S^{ai}_u.$$

Applying the Fatou lemma for conditional expectations, we get (3.1).

Let us now prove that

$$\mathsf{E}_{\mathsf{Q}}(S^{ai}_v \mid \mathcal{F}_u) \ge S^{bi}_u. \tag{3.3}$$

For u_m, v_n defined above and any $D \in \mathcal{F}_{u_m}$, we have

$$\mathsf{E}_{\mathsf{Q}} I_D(-S^{ai}_{v_n} + S^{bi}_{u_m}) \le 0$$

(recall that $S^{ai} \le \gamma^i S^{bi}$). Thus,

$$\mathsf{E}_{\mathsf{Q}}(S^{ai}_{v_n} \mid \mathcal{F}_{u_m}) \ge S^{bi}_{u_m}.$$

Arguing in the same way as above, we get

$$\mathsf{E}_{\mathsf{Q}}(S^{ai}_{v_n} \mid \mathcal{F}_u) \ge S^{bi}_u. \tag{3.4}$$

It follows that, for any $(\mathcal{F}_t)$-stopping time v,

$$S^{ai}_v \le \gamma^i S^{bi}_v \le \gamma^i \mathsf{E}_{\mathsf{Q}}(S^{ai}_T \mid \mathcal{F}_v) \le (\gamma^i)^2 \mathsf{E}_{\mathsf{Q}}(S^{bi}_T \mid \mathcal{F}_v).$$

Using the inclusion $\mathsf{Q} \in \mathcal{R}(Z_0)$, it is easy to check that S^{bi}_T is Q-integrable, and hence, the collection $(S^{ai}_{v_n})_{n=1}^{\infty}$ is Q-uniformly integrable. Now, (3.3) follows from (3.4).

Consider the Snell envelopes

$$X_t = \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \mathsf{E}_{\mathsf{Q}}(S^{bi}_\tau \mid \mathcal{F}_t), \quad t \in [0,T],$$
$$Y_t = \operatorname*{essinf}_{\tau\in\mathcal{T}_t} \mathsf{E}_{\mathsf{Q}}(S^{ai}_\tau \mid \mathcal{F}_t), \quad t \in [0,T],$$

where $\mathcal{T}_t$ denotes the set of $(\mathcal{F}_t)$-stopping times such that $\tau \ge t$. (Recall that $\operatorname{esssup}_\alpha \xi_\alpha$ is a random variable ξ such that, for any α, $\xi \ge \xi_\alpha$ a.s. and for any other random variable ξ' with this property, we have $\xi \le \xi'$ a.s.) Then X is an $(\mathcal{F}_t, \mathsf{Q})$-supermartingale, while Y is an $(\mathcal{F}_t, \mathsf{Q})$-submartingale (see [7; Th. 2.12.1]).

Let us prove that, for any $t \in [0,T]$, $X_t \le Y_t$ Q-a.s. Assume that there exists t such that $\mathsf{P}(X_t > Y_t) > 0$. Then there exist $\tau, \sigma \in \mathcal{T}_t$ such that

$$\mathsf{Q}\big(\mathsf{E}_{\mathsf{Q}}(S^{bi}_\tau \mid \mathcal{F}_t) > \mathsf{E}_{\mathsf{Q}}(S^{ai}_\sigma \mid \mathcal{F}_t)\big) > 0.$$

This implies that $\mathsf{Q}(\xi > \eta) > 0$, where $\xi = \mathsf{E}_\mathsf{Q}(S^{bi}_\tau \mid \mathcal{F}_{\tau\wedge\sigma})$ and $\eta = \mathsf{E}_\mathsf{Q}(S^{ai}_\sigma \mid \mathcal{F}_{\tau\wedge\sigma})$. Assume first that $\mathsf{Q}(\{\xi > \eta\} \cap \{\tau \le \sigma\}) > 0$. On the set $\{\tau \le \sigma\}$, we have

$$\xi = S^{bi}_\tau = S^{bi}_{\tau\wedge\sigma},$$
$$\eta = \mathsf{E}_\mathsf{Q}(S^{ai}_\sigma \mid \mathcal{F}_{\tau\wedge\sigma}) = \mathsf{E}_\mathsf{Q}(S^{ai}_{\tau\wedge\sigma} \mid \mathcal{F}_{\tau\wedge\sigma}),$$

and we obtain a contradiction with (3.3). Similarly, if we assume that $\mathsf{Q}(\{\xi > \eta\} \cap \{\tau \ge \sigma\}) > 0$, then we arrive at a contradiction with (3.1). As a result, $X_t \le Y_t$ Q-a.s. Now, an application of Lemma 3.2 shows that $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Then, for each i, there exists an $(\mathcal{F}_t, \mathsf{Q})$-martingale M^i such that, for any $t \in [0,T]$, $S^{bi}_t \le M^i_t \le S^{ai}_t$ Q-a.s. Fix

$$X = \sum_{n=0}^{N} \sum_{i\in I} \big[-H^i_n I(H^i_n > 0) S^{ai}_{u_n} - H^i_n I(H^i_n < 0) S^{bi}_{u_n}\big] \in A.$$

Let $(\widetilde{\mathcal{F}}_t)$ denote the Q-completion of $(\mathcal{F}_t)$. The process M^i admits a càdlàg $(\widetilde{\mathcal{F}}_t)$-modification $\widetilde{M}^i$ (see [21; Ch. II, Th. 2.9]). We have

$$\begin{aligned} X &\le \sum_{n=0}^{N} \sum_{i\in I} \big[-H^i_n I(H^i_n > 0) \widetilde{M}^i_{u_n} - H^i_n I(H^i_n < 0) \widetilde{M}^i_{u_n}\big] \\ &= \sum_{n=1}^{N} \sum_{i\in I} \bigg[\bigg(\sum_{k=0}^{n-1} H^i_k\bigg)\big(\widetilde{M}^i_{u_n} - \widetilde{M}^i_{u_{n-1}}\big)\bigg]. \end{aligned}$$

The process

$$M_l = \sum_{n=1}^{l} \sum_{i\in I} \bigg[\bigg(\sum_{k=0}^{n-1} H^i_k\bigg)\big(\widetilde{M}^i_{u_n} - \widetilde{M}^i_{u_{n-1}}\big)\bigg], \quad l = 0, \dots, N$$

is a Q-local martingale with respect to the filtration $(\mathcal{F}_{u_l})$. Now, it follows from Lemma 3.3 that $\mathsf{E}_\mathsf{Q} M_N^- \ge \mathsf{E}_\mathsf{Q} M_N^+$. Consequently, $\mathsf{E}_\mathsf{Q} X^- \ge \mathsf{E}_\mathsf{Q} X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

Let us now consider a model with proportional transaction costs, i.e. a model with $S^{bi} = (1-\lambda^i) S^{ai}$, where $\lambda^i \in [0,1)$ is the coefficient of proportional transaction costs for the i-th asset. We introduce the following definition.

Definition 3.4. An $\mathbb{R}_+$-valued process X is called an $(\mathcal{F}_t, \mathsf{P})$-*delta-martingale of order* a, where $a \in (0,1]$ if

(a) X is $(\mathcal{F}_t)$-adapted and càdlàg;
(b) $\mathsf{E} X_t < \infty$, $t \in [0,T]$;
(c) for any $(\mathcal{F}_t)$-stopping times $u \le v$, we have $a X_u \le \mathsf{E}(X_v \mid \mathcal{F}_u) \le a^{-1} X_u$.

It is seen from the proof of Key Lemma 3.1 that X is a delta-martingale of order a if and only if there exists a martingale M such that, for any $t \in [0,T]$, $a X_t \le M_t \le X_t$ a.s. Consequently, in models with proportional transaction costs

$$\mathcal{R} = \{\mathsf{Q} \sim \mathsf{P} : \text{for any } i \in I,\ S^{ai} \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-delta-martingale of order } 1-\lambda^i\}. \quad (3.5)$$

Let us now study the following problem. Consider a one-dimensional model with the coefficient λ of proportional transaction costs and denote by $I_\lambda(F)$ the fair price interval in this model. *Is it true that* $I_\lambda(F) \xrightarrow[\lambda\downarrow 0]{} I(F)$, *where* $I(F)$ *is the fair price interval in the model with no transaction costs* (*i.e. with* $\lambda = 0$)? This problem was considered for the Black–Scholes model in the papers [5], [20], [24], and it was proved that the upper price of a European call option $(S_T - K)^+$ tends to S_0 as $\lambda \downarrow 0$. Our approach to arbitrage pricing in models with transaction costs is different from the one in the papers mentioned, but the same result turns out to be true in our approach as well. Thus, the answer to the question posed above is negative for natural continuous-time models.

Proposition 3.5. *Let* $S_t = S_0 e^{\mu t + \sigma B_t}$, *where* $\mu \in \mathbb{R}$, $\sigma > 0$, *and* B *is a Brownian motion. Let* $\mathcal{F}_t = \mathcal{F}_t^B$, $S^a = S$, $S^b = (1-\lambda)S$, $F = (S_T - K)^+$. *Then*

$$I_\lambda(F) \xrightarrow[\lambda\downarrow 0]{} ((S_0 - K)^+, S_0)$$

in the sense that the left (*resp., right*) *endpoints of* $I_\lambda(F)$ *tend to* $(S_0 - K)^+$ (*resp.,* S_0) *as* $\lambda \downarrow 0$.

`Proof`. It is clear that $I_\lambda(F)$ decreases as $\lambda \downarrow 0$. Furthermore, using static considerations (i.e. considering trades at dates 0 and T only), one can easily see that, for any $\varepsilon > 0$, there exists $\lambda > 0$ such that $I_\lambda(F) \subseteq ((S_0 - K)^+ - \varepsilon, S_0 + \varepsilon)$. Thus, it will suffice to prove that, for any $\lambda > 0$,

$$I_\lambda(F) \supseteq ((S_0 - K)^+, S_0). \tag{3.6}$$

Clearly, we can assume from the outset that $\Omega = C([0,T])$, $S = X$, where X denotes the coordinate process (i.e. $X_t(\omega) = \omega(t)$), and $\mathcal{F}_t = \mathcal{F}_t^X$.

Step 1. Let $a > 0$ and Z be a solution of the stochastic differential equation

$$dZ_t = -a \operatorname{sgn}(Z_t - S_0) I(t \le \tau) dt + \sigma Z_t dW_t, \quad Z_0 = S_0,$$

where $\tau = \inf\{t \ge 0 : |Z_t - S_0| \ge \Delta\}$, $\Delta = \lambda S_0/10$, and W is a Brownian motion. The process Z is defined on a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathsf{P}})$. Fix $(\widetilde{\mathcal{F}}_t)$-stopping times $u \le v$. On the set $\{\tau \le u\}$, we have $\mathsf{E}_{\widetilde{\mathsf{P}}}(Z_v \mid \widetilde{\mathcal{F}}_u) = Z_u$. On the set $\{\tau > u\}$, we have $\mathsf{E}_{\widetilde{\mathsf{P}}}(Z_v \mid \widetilde{\mathcal{F}}_u) = \mathsf{E}_{\widetilde{\mathsf{P}}}(Z_{v\wedge\tau} \mid \widetilde{\mathcal{F}}_u)$, $|Z_u - S_0| \le \Delta$, and $|Z_{v\wedge\tau} - S_0| \le \Delta$. Thus,

$$\mathsf{E}_{\widetilde{\mathsf{P}}}(Z_v \mid \widetilde{\mathcal{F}}_u) \ge (1-\lambda) Z_u, \qquad \mathsf{E}_{\widetilde{\mathsf{P}}}((1-\lambda) Z_v \mid \widetilde{\mathcal{F}}_u) \le Z_u.$$

Now, set $\mathsf{Q}(a) = \operatorname{Law}(Z_t; t \le T)$. Then, for any $(\mathcal{F}_t)$-stopping times $u \le v$, we have

$$\mathsf{E}_{\mathsf{Q}(a)}(X_v \mid \mathcal{F}_u) \ge (1-\lambda) X_u, \qquad \mathsf{E}_{\mathsf{Q}(a)}((1-\lambda) X_v \mid \mathcal{F}_u) \le X_u.$$

Furthermore, Girsanov's theorem guarantees that $\mathsf{Q}(a) \sim \mathsf{P}$. In view of (3.5), $\mathsf{Q}(a)$ is a risk-neutral measure.

Let us prove that

$$\lim_{a\to\infty} \mathsf{E}_{\mathsf{Q}(a)} F = (S_0 - K)^+. \tag{3.7}$$

For any $b > 2S_0$, we have, by the Itô-Tanaka-Meyer formula,

$$(Z_t - b)^+ = \int_0^t I(Z_s > b) \sigma Z_s dW_s + \frac{1}{2} L_t^b(Z), \quad t \ge 0,$$

where $L_t^b(Z)$ denotes the local time spent by the process Z at the point b by the time t. It follows from this representation that $\mathsf{E}(Z_T - b)^+ \le \mathsf{E}(Z_\sigma - b)^+$, where

$\sigma = TI(\tau \geq T) + (T+\tau)I(\tau < T)$. Using this inequality and the property that $(Z_T - b)^+ = 0$ on $\{\tau \leq T\}$, we can write

$$\begin{aligned}\mathsf{E}(Z_T - b)^+ &\leq \mathsf{E}(Z_{T+\tau} - b)^+ I(\tau < T, Z_\tau = S_0 - \Delta) + \mathsf{E}(Z_{T+\tau} - b)^+ I(\tau < T, Z_\tau = S_0 + \Delta)\\ &= \mathsf{E}\big(Y_T^{S_0-\Delta} - b\big)^+ \mathsf{P}(\tau < T, Z_\tau = S_0 - \Delta) + \mathsf{E}\big(Y_T^{S_0+\Delta} - b\big)^+ \mathsf{P}(\tau < T, Z_\tau = S_0 + \Delta),\end{aligned}$$

where Y^x is a solution of the stochastic differential equation

$$dY_t^x = \sigma Y_t^x dW_t, \quad Y_0^x = x.$$

It is seen from the inequality proved above that $\mathsf{E}(Z_T - b)^+$ converges to 0 uniformly in $a > 0$ as $b \to \infty$. Furthermore, it is clear that Z_T converge weakly to S_0 as $a \to \infty$. This yields (3.7).

Step 2. Let $a, b, c > 0$ and $(Z, \widetilde{Z})$ be a solution of the system

$$\begin{aligned}dZ_t &= -a\,\mathrm{sgn}(Z_t - \widetilde{Z}_t)I(t \leq \tau)dt + \sigma Z_t dW_t, \quad Z_0 = S_0,\\ d\widetilde{Z}_t &= b\widetilde{Z}_t d\widetilde{W}_t, \quad \widetilde{Z}_0 = S_0,\end{aligned}$$

where $\tau = \inf\{t \geq 0 : |Z_t - \widetilde{Z}_t| \geq \Delta \text{ or } Z_t \leq c\}$, $\Delta = \lambda S_0/10$, and W, $\widetilde{W}$ are independent Brownian motions. The process $(Z, \widetilde{Z})$ is defined on a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathsf{P}})$. Arguing in the same way as above, we check that, for any $(\widetilde{\mathcal{F}}_t)$-stopping times $u \leq v$,

$$\mathsf{E}_{\widetilde{\mathsf{P}}}(Z_v \mid \widetilde{\mathcal{F}}_u) \geq (1-\lambda)Z_u, \qquad \mathsf{E}_{\widetilde{\mathsf{P}}}((1-\lambda)Z_v \mid \widetilde{\mathcal{F}}_u) \leq Z_u$$

provided that $c < S_0$. Hence, the measure $\mathsf{Q}(a,b,c) = \mathrm{Law}(Z_t;\, t \leq T)$ is a risk-neutral measure. Clearly,

$$\lim_{c\downarrow 0}\lim_{a\to\infty} \mathrm{Law}(Z_t;\, t \leq T) = \mathrm{Law}(\widetilde{Z}_t;\, t \leq T),$$

and therefore,

$$\liminf_{b\to\infty}\liminf_{c\downarrow 0}\liminf_{a\to\infty} \mathsf{E}_{\mathsf{Q}(a,b,c)}F \geq \liminf_{b\to\infty} \mathsf{E}(\widetilde{Z}_T - K)^+ = \lim_{b\to\infty} \mathsf{E}(\widetilde{Z}_T - K)^+ = S_0. \tag{3.8}$$

Relations (3.7) and (3.8) taken together yield (3.6). □

4 Model with European Call Options as Basic Assets

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $T \in [0, \infty)$. Let S_T be an $\mathbb{R}_+$-valued random variable. From the financial point of view, S_T is the ask price of some asset at time T. For simplicity, we consider only proportional transaction costs on the underlying assets, i.e. the bid price of the i-th asset at time T is $(1-\lambda)S_T$, where $\lambda \in [0,1)$. Let $\mathbb{K} \subseteq \mathbb{R}_+$ be the set of strike prices K of traded European call options on this asset with maturity T. Let $\varphi^a(K)$ and $\varphi^b(K)$, $K \in \mathbb{K}$ be the ask and bid prices at time 0 of such an option. Define the set of attainable incomes by

$$A = \bigg\{\sum_{n=1}^{N}[g_n(((1-\lambda)S_T - K_n)^+ - \varphi^a(K_n)) + h_n(-(S_T - K_n)^+ + \varphi^b(K_n))] : \\ N \in \mathbb{N},\ K_n \in \mathbb{K},\ g_n, h_n \in \mathbb{R}_+\bigg\}.$$

We assume that $0 \in \mathbb{K}$, which means the possibility to trade the underlying asset.

`Notation`. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \mathrm{Law}_{\mathsf{Q}}\, S_T \in \mathcal{D}\},$$

where

$$\mathcal{D} = \{\varphi'' : \varphi \text{ is convex on } \mathbb{R}_+,\ \varphi'_+(0) \geq -1,\ \lim_{x\to\infty} \varphi(x) = 0,$$
$$\varphi((1-\lambda)^{-1}K) \leq (1-\lambda)^{-1}\varphi^a(K) \text{ and } \varphi(K) \geq \varphi^b(K),\ K \in \mathbb{K}\}.$$

Here φ'_+ denotes the right-hand derivative and φ'' denotes the second derivative taken in the sense of distributions (i.e. $\varphi''((a,b]) = \varphi'_+(b) - \varphi'_+(a)$) with the convention: $\varphi''(\{0\}) = \varphi'_+(0) + 1$ (thus, φ'' is a probability measure provided that $\varphi'_+(0) \geq -1$).

Key Lemma 4.1. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$*, we have*

$$\mathcal{R} = \mathcal{R}((1-\lambda)S_T - \varphi^a(0)) = \mathcal{M}.$$

`Proof`. Denote $(1-\lambda)S_T - \varphi^a(0)$ by Z_0.

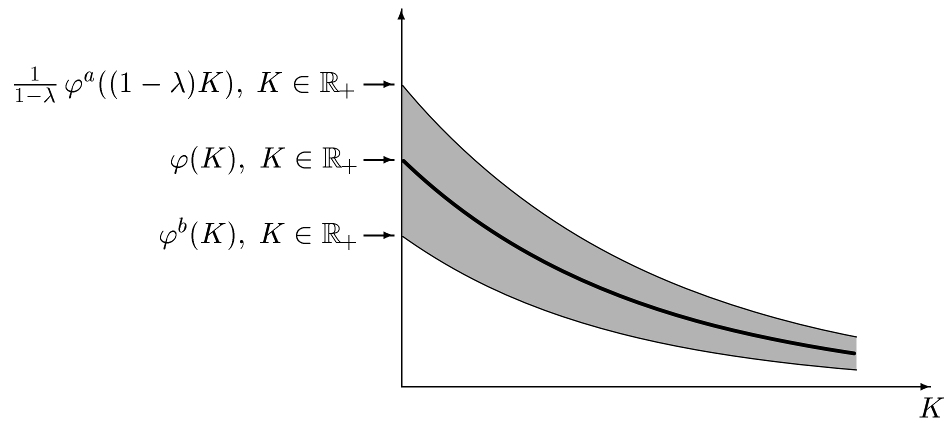
Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 2.4.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Fix $\mathsf{Q} \in \mathcal{R}(Z_0)$. By considering the function $\varphi(x) = \mathsf{E}_{\mathsf{Q}}(S_T - x)^+$, we conclude that $\mathsf{Q} \in \mathcal{M}$.

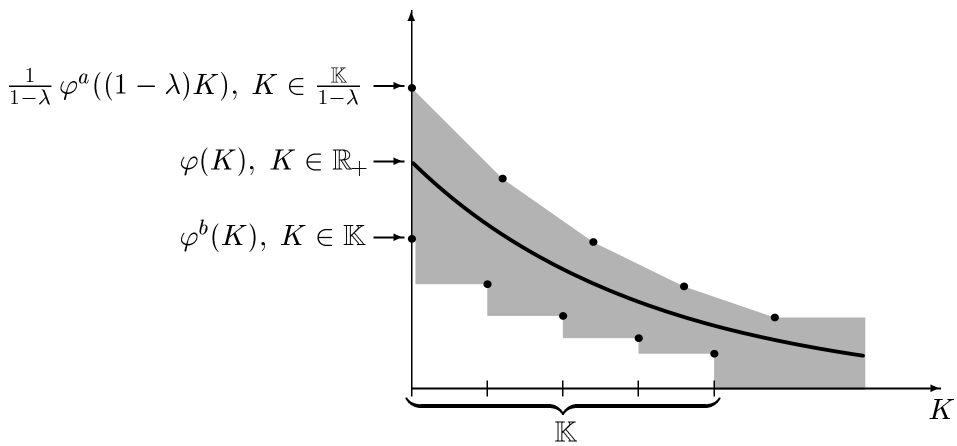
Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Fix $\mathsf{Q} \in \mathcal{M}$ with $\mathrm{Law}_{\mathsf{Q}}\, S_T = \varphi''$. Then

$$\mathsf{E}_{\mathsf{Q}}(S_T - K)^+ = \int_{\mathbb{R}_+} (x-K)^+ \varphi''(dx) = \varphi(K), \quad K \in \mathbb{R}_+.$$

Consequently, $\mathsf{E}_{\mathsf{Q}} X \leq 0$ for any $X \in A$, which means that $\mathsf{Q} \in \mathcal{R}$. □



`Figure 1.a.` The structure of $\mathcal{D}$ in the case, where $\mathbb{K} = \mathbb{R}_+$. The set $\mathcal{D}$ consists of the second derivatives φ'', where φ is convex on $\mathbb{R}_+$, $\varphi'_+(0) \geq -1$, $\lim\limits_{x\to\infty} \varphi(x) = 0$, and φ lies in the shaded region.



`Figure 1.b.` The structure of $\mathcal{D}$ in the case, where $\mathbb{K}$ is finite. The set $\mathcal{D}$ consists of the second derivatives φ'', where φ is convex on $\mathbb{R}_+$, $\varphi'_+(0) \geq -1$, $\lim\limits_{x\to\infty} \varphi(x) = 0$, and φ lies in the shaded region.

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